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A Theory of Truthmaker Content II: Subject-matter, Common Content, Remainder and Ground

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We continue with the development of the theory of truthmaker content begun in part I, dealing with such 'non-standard' topics as subject matter, common content, logical remainder and ground. This is by no means an exhaustive list of topics that might have been considered but it does provide an indication of the nature and scope of the theory. As before, the paper is divided into an informal exposition and a technical addendum. Both can be read independently of the other but it would be helpful, in either case, to have the first part of the paper at hand.

One feature of great interest in the present account is that it deals with a number of the topics that lack an adequate treatment either within the possible worlds account or under a structural conception of propositions. The notion of common content, for example, can be readily handled within the present framework but cannot be properly handled in the other two frameworks without either introducing or deriving something like the present conception of verification. Thus we should not simply regard the present 'extensional' conception of content as a mere approximation to a structural conception but as an important conception in its own right.

Another feature of interest is that many of the notions we shall consider are obtained by 'lifting' corresponding notions at the level of verifying states to the level of propositions. Take again the notion of common content. There is a notion of common part for states, what it is they have in common; and this may then be extended from states to propositions. Likewise for the notion of remainder. We may

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subtract one state from another; and, similarly, we may consider the result of subtracting one proposition from another. This process of lifting gives the theory a familiar mathematical feel while also providing a firm intuitive foundation for the derived notions.

A final feature of general interest concerns duality. We are all familiar with the notion of duality in classical logic; disjunction, for example, is dual to conjunction and will therefore behave in a related way. We may think of the duality in classical logic as arising from the possibility of 'reversing' the relation of classical consequence. Thus if a conjunction classically entails its conjuncts then a disjunction will be classically entailed by its disjuncts. Conjunction and disjunction are not dual in this way within the present framework. But they are dual in a different way. Instead of replacing the relation of conjunctive part (or containment) by its converse, we replace it by the notion of disjunctive part (or entailment). Thus whereas a conjunction will 'contain' its conjuncts, a disjunction will be 'entailed' by its disjuncts. We might call this 'horizontal' duality. But there will also be a 'vertical' duality of the familiar sort. Thus in the case of both containment and entailment we may consider the notions obtained by reversing these relations. We thereby obtain a much richer theory, in which the two aspects of duality operate independently of one another.

Let us now give an informal exposition of the topics of interest to us before proceeding to the more formal exposition.

1 Aboutness

I have so far talked mainly of partial content. But there is a slew of related concepts to which the present methods are also applicable. Partial content is merely the tip of the iceberg; and many of the considerations previously raised against the intensional approach to partial content perhaps apply with even greater force in these other cases.

One such concept is that of *subject-matter*, of what a proposition is about.[1] I previously remarked that every regular verifiable proposition will have a maximal verifier; and I should now like to suggest that the subject-matter of a proposition (or, at least, of a regular proposition) be identified with its maximal verifier. Thus, where P is the unilateral proposition $\{p_1, p_2, ...\}$, its subject-matter, denoted by $\boldsymbol{p}$, will be the fusion $p_1 \sqcup p_2 \sqcup ...$ of all its verifiers.

This might appear to be a surprising, even an absurd, thing to suggest. For consider the proposition that it does or does not rain and the proposition that it is or is not snowing. The subject-matter of the first proposition is the fusion of the presence and absence of rain, let us say, while the subject-matter of the second proposition is the fusion of the presence and absence of snow. But these are identical impossible states and so the subject-matter of the two propositions is the same and we lose the fact that the one proposition concerns rain while the other concerns snow.

But this line of reasoning rests upon adopting a too coarse-grained conception of impossible states, under which any two of them will be the same. Suppose, on

[1] Further discussion of subject-matter is to be found in [7].

the other hand, that we adopt a more fine-grained conception of impossible states - one that allows us, in effect, to recover the possible states from which they were composed. Then the two impossible states will differ in that the one is composed of the presence and absence of rain while the other is composed of the presence and absence of snow, without the component parts of either being parts of the other. I have shown in [6] how to construct such a fine-grained ontology of impossible states from an underlying ontology of possible states and by appeal to this work or by simply taking suitably fine-grained impossible states as given, the present objection may be met.[2]

Given such an ontology of states, we may regard 'logical space', the space from which the verifiers of all propositions are drawn, as the full state $\blacksquare$. A subject-matter is then simply the part of logical space that is obtained by distinguishing and then combining the particular facts or states of concern to us. We are used to thinking of logical space as a set of worlds, standing in grand isolation from one another, and so the idea of combining incompatible parts of different worlds makes little sense. But once we adopt this more comprehensive view, we can locate subject-matter where it belongs, at the level of the facts themselves.

Once subject-matters are identified with states, the relation of part-whole on subject-matters can be identified with the relation of part-whole on states and we can talk of mereological relationships among subject-matters in the same straightforward way and subject to the same general principles as talk of mereological relationships among states. Thus we get for 'free' what it is for one subject-matter to be part of another, or for two subject-matters to have a given common part, or for two subject-matters to combine into a more comprehensive subject-matter.

It may be helpful to compare the present account of subject-matter to that developed by Lewis [1988] within the context of the possible worlds semantics. For Lewis, a subject-matter is given by an equivalence relation on worlds. Intuitively, two worlds will stand in the equivalence relation when they do not differ with regard to the subject-matter. Thus if the subject-matter is the current weather in New York, then two worlds will stand in the associated equivalence relation when they do not differ with regard to the current state of the weather in New York.

To each subject-matter $\boldsymbol{s}$ in my sense will correspond an equivalence relation on worlds. For we may take two worlds w and v to be equivalent when their intersections $w \sqcap \boldsymbol{s}$ and $v \sqcap \boldsymbol{s}$ with the subject-matter $\boldsymbol{s}$ are the same. But the converse is not necessarily true, there may be equivalence relations on worlds that do not correspond to subject-matters in my sense (indeed, this will clearly be the case when there are more than two worlds and they have no proper non-null parts); and nor do distinct subject-matters in my sense necessarily correspond to distinct equivalence relations on worlds (as is clear when the subject-matters are constituted by distinct necessary states). Thus Lewis can make distinctions in subject-matter that I cannot capture and I can make distinctions in subject-matter that he cannot capture. What to make of this mismatch is hard to say, but my own conception is, in many ways, much better suited

[2] I also make use of a variety of impossible states in developing a truthmaker semantics for intuitionistic logic in [4].

to developing a theory of the relationship between propositions and their subject matter.

For one thing, as pointed out in [12], Section 1.10, Lewis does not have a particularly good conception of *the* subject-matter of a proposition. For a proposition to be about some subject-matter, as given by an equivalence relation on worlds, it should not discriminate between equivalent worlds, being true in some and not in others. But of all the subject-matters that a proposition is about in this sense, from the very finest to the very crudest, none seems to stand out as *the* subject-matter.

But there is a deeper problem. For there is serious tension between the intensional approach adopted by Lewis and what we might reasonably expect of an adequate account of subject-matter. Let me illustrate with a notion of subject-matter for which the problem of identifying a *particular* subject-matter does not arise. This is the notion of *being partly about*, which Lewis discusses at some length from within the intensional framework. There are, of course, many subject-matters which a given proposition is partly about but none which it is especially partly about.

Given some subject-matter $\boldsymbol{s}$, then a definition of what it is for a proposition P to be partly about $\boldsymbol{s}$ immediately suggests itself within our own framework. It is for the subject-matter $\boldsymbol{p}$ of P and the subject-matter $\boldsymbol{s}$ to have a common (non-null) part.[3] Thus the proposition that the Absolute is sleeping in the rain or is awake will partly be about the weather (a subject of much greater interest than the Absolute in most parts of the world), since its subject-matter will contain a part, the rain, in common with the weather (construed as the fusion of all states of the weather).

Not only does this account deliver the right judgements in particular cases, it also conforms to some highly plausible principles:

Strong Composition if P is partly about $\boldsymbol{s}$ then so are $P \vee Q$ and $P \wedge Q$; and if P is about $\boldsymbol{s}$ then so is $\neg P$;[4]

Weak Entailment Any proposition containing a proposition partly about $\boldsymbol{s}$ is partly about $\boldsymbol{s}$.

Compare this now with the discussion in Lewis [1988]. He formulates a somewhat weaker compositional principle:

Weak Composition if P and Q are both partly about $\boldsymbol{s}$ then so are $\neg P$, $P \vee Q$ and $P \wedge Q$ as long as they are not 'analytic' (or necessary);

and a somewhat stronger entailment principle:

Strong Entailment Any proposition necessitating a proposition partly about $\boldsymbol{s}$ is partly about $\boldsymbol{s}$.

He then points out that these principles will lead to 'collapse'; any proposition P will be partly about $\boldsymbol{s}$. For suppose, as is reasonable, that there is at least one proposition Q partly about $\boldsymbol{s}$. Then $P \wedge Q$ and $P \wedge \neg Q$ will be partly about $\boldsymbol{s}$ by Strong Entailment; so $(P \wedge Q) \vee (P \wedge \neg Q)$ will be partly about $\boldsymbol{s}$ by Weak Composition; and

[3] We should perhaps also allow any proposition P to be partly about the null subject-matter $\square$.

[4] Closure under negation also requires that we take the falsifiers of a proposition into account in ascertaining its subject-matter.

so, given that $(P \wedge Q) \vee (P \wedge \neg Q)$ is necessarily equivalent to P, P will be partly about $\boldsymbol{s}$ by Strong Entailment again. He therefore suggests that the two principles are fundamentally in conflict and goes on to propose different ways in which the conflict might be resolved.

We should notice right away something bizarre about his presentation of the dialectical situation. For in so far as Lewis' weaker compositional principle has any plausibility for the notion of being partly about then so does our stronger principle. If, for example, one is tempted to say that $P \wedge Q$ is partly about $\boldsymbol{s}$, given that both P and Q are partly about $\boldsymbol{s}$, then should one not be equally tempted to say that $P \wedge Q$ is partly about $\boldsymbol{s}$ if either P or Q is about $\boldsymbol{s}$? Not so, of course, for being *entirely* about, but what is here in question is being *partly* about.

But once one has the stronger compositional principle, then one does not need anything so strong as the entailment principle to get collapse. All one needs is:

Equivalence Any proposition necessarily equivalent to a proposition partly about $\boldsymbol{s}$ is partly about $\boldsymbol{s}$.

For suppose that P necessitates a proposition Q partly about $\boldsymbol{s}$. Then $P \wedge Q$ is partly about $\boldsymbol{s}$ by Strong Composition; and, given that P is necessarily equivalent to $P \wedge Q$, P is partly about $\boldsymbol{s}$ by Equivalence. Thus Equivalence, in the presence of Strong Composition, will imply Strong Entailment; and the previous demonstration of collapse will go through.

This suggests that the true conflict is not between Weak Composition and Strong Entailment but between Strong Composition and Equivalence. Once given Equivalence, we can no longer hold on to a reasonable compositional account of subject-matter. Of course, Lewis prefers not to see things this way since he wants (for the most part) to hold onto a possible worlds account of content. But this then leaves us with a seriously gerry-mandered and compromised conception of what it is.

Once we are open to giving up Equivalence, we can accept Composition in full force and not just some artificially weakened version of Composition. We can also see, within the present conception of truthmaker content, what was right about Equivalence and Strong Entailment, even though both of them should be given up. For P will be partly about what Q is partly about if it *contains* Q. Indeed, how could this fail to be so if the content Q is part of the content P? Thus Strong Entailment arises from the confused thought that a proposition will contain any proposition that it entails. I might further note that there is no need to make an exception to Composition in the case of 'analytic' propositions. For $P \vee \neg P$ will not in general contain or be contained in $Q \vee \neg Q$ and so there is no danger, given that $P \vee \neg P$ is partly about what P is about (through the application of Composition), that $Q \vee \neg Q$ will then be partly about what P is partly about (through an appropriate application of Equivalence).

I suspect that it has often been thought, if only implicitly, that a compositionally adequate account of subject-matter could only be achieved by adopting a structural conception of propositions, under which the subject-matter of a proposition is somehow discerned from its structure. Adopt the more usual 'extensional' conception of propositions, under which a proposition is identified with a set of verifiers, and some compositional principles must perforce be given up. Hence perhaps, Lewis' equanimity in accepting a weakened version of Composition.

The present truthmaker account of propositions shows this line of thought to be mistaken. Once relevance is built into the very conception of verification, the notion of subject-matter will be reasonably well-behaved. The defects and distortions of the intensional approach arise, not from its embrace of an extensional conception of content, but from its indifference to considerations of relevance.

2 Round Aboutness

There are a number of other concepts related to subject-matter which may be readily defined within the present framework.

We may say, for example, that a proposition P is *entirely about* the subject-matter $\boldsymbol{s}$ if the subject-matter $\boldsymbol{p}$ of P is part of $\boldsymbol{s}$. Thus the proposition that it is snowing or raining is entirely about the weather since the verifiers of the proposition, snow or rain, are part of the weather. More generally, once we have a conception of the subject-matter $\boldsymbol{p}$ of a proposition, we may define different relationships of a proposition to a subject-matter $\boldsymbol{s}$ in terms of the relationship of $\boldsymbol{s}$ to $\boldsymbol{p}$. Thus:

P is *exactly about* $\boldsymbol{s}$ if $\boldsymbol{p} = \boldsymbol{s}$;

P is *partly about* $\boldsymbol{s}$ if $\boldsymbol{p}$ and $\boldsymbol{s}$ overlap

P is *entirely about* $\boldsymbol{s}$ if $\boldsymbol{p} \subseteq \boldsymbol{s}$

P is *about* $\boldsymbol{s}$ *in its entirety* if $\boldsymbol{p} \supseteq \boldsymbol{s}$.

We can also get at the idea that one proposition is the restriction of the content of another proposition to some given subject-matter. Intuitively, the restricted content is that part of the original content that is exactly about the given subject-matter. For example, when P is the proposition that the Absolute is walking in the rain or lounging in the sun and when the subject matter $\boldsymbol{s}$ is the weather, then the restriction of P to $\boldsymbol{s}$ will be the proposition that it is rainy or sunny. Of more interest to philosophers are cases in which the restriction is to some very general features of logical space, such as the observable or the concrete or the actual. We obtain in this way the idea of the observational content or the nominalistic content or the true content of a given proposition (something that was an impetus for Lewis' account of subject-matter).

We may define the general concept of propositional restriction as follows. Given a proposition P and subject-matter $\boldsymbol{s}$, we take the *restriction of P to* $\boldsymbol{s}$ to consist of the common parts of the verifiers of P and the subject-matter $\boldsymbol{s}$. Thus when P is the proposition $\{p_1, p_2, ...\}$, its restriction to $\boldsymbol{s}$ will be $\{p_1 \sqcup \boldsymbol{s}, p_2 \sqcup \boldsymbol{s}, ...\}$. In the example above, the common part of the Absolute walking in the rain and the weather will be the presence of rain and the common part of the Absolute lounging in the sun and the weather will be the presence of sun, so that the restriction of the proposition P to the weather will be the proposition that is verified either by the presence of rain or the presence of sun.

It is perhaps worth pointing out a certain difficulty in defining the restriction of a *bilateral* proposition (one constituted by a set of verifiers and a set of falsifiers). Here are three natural constraints on the restriction of a bilateral proposition to some given subject-matter $\boldsymbol{s}$:

(1) If $\boldsymbol{P}$ is entirely about $\boldsymbol{s}$ and $\boldsymbol{Q}$ is not at all about $\boldsymbol{s}$ then the restriction of $\boldsymbol{P} \wedge \boldsymbol{Q}$ to $\boldsymbol{s}$ is $\boldsymbol{P}$;
(2) The restriction of $\neg\boldsymbol{P}$ to $\boldsymbol{s}$ is the negation of the restriction of $\boldsymbol{P}$ to $\boldsymbol{s}$;
(3) The restriction of $\boldsymbol{P}$ to $\boldsymbol{s}$ is entailed by $\boldsymbol{P}$.

But all three constraints cannot be satisfied. For let $\boldsymbol{P}$ be the proposition that it is cold in London, $\boldsymbol{Q}$ the proposition that it is cold in New York, and $\boldsymbol{s}$ the subject of the temperature in London (a common topic of conversation there). Then the antecedent of (1) is satisfied; and so the restriction of $\boldsymbol{P} \wedge \boldsymbol{Q}$ to $\boldsymbol{s}$ is $\boldsymbol{P}$. By (2), the restriction of $\neg(\boldsymbol{P} \wedge \boldsymbol{Q})$ to $\boldsymbol{s}$ is $\neg\boldsymbol{P}$. But $\neg(\boldsymbol{P} \wedge \boldsymbol{Q})$ does not entail $\neg\boldsymbol{P}$, contrary to (3).

I have stated this problem as a difficulty within my own theory, but it is a general difficulty for any account of propositional restriction. Conditions (1) and (3) are presumably non-negotiable; and so the difficulty shows that any reasonable account of propositional restriction must distinguish between the restriction of a negation and the negation of a restriction.

Within the possible worlds framework, for example, the restriction of a proposition $\boldsymbol{P}$ to some subject-matter $\boldsymbol{s}$ might be taken to be the set of worlds $\boldsymbol{s}$-equivalent to some world in $\boldsymbol{P}$, where $\boldsymbol{s}$-equivalence between worlds is a matter of their agreement with respect to $\boldsymbol{s}$. The restriction of $\boldsymbol{P} \wedge \boldsymbol{Q}$ to $\boldsymbol{s}$ in the above example will then be $\boldsymbol{P}$, but the restriction of $\neg(\boldsymbol{P} \wedge \boldsymbol{Q})$ to $\boldsymbol{s}$ will be the set of all worlds, since presumably each world in which London is cold will be $\boldsymbol{s}$-equivalent, i.e. have the same London temperature, as some world in which New York is not cold.

Our own framework provides a somewhat more satisfactory resolution of the difficulty. For suppose that $\boldsymbol{P}$ is falsified by p' (the fact, say, that it is warm in London) and $\boldsymbol{Q}$ is falsified by q' (the fact that it is warm in New York). Then the restriction of $\neg(\boldsymbol{P} \wedge \boldsymbol{Q})$ to $\boldsymbol{s}$ will be verified by the null state (since q' and $\boldsymbol{s}$ have nothing in common), but also by p' (since p' is a part of $\boldsymbol{s}$); and so we can at least be assured that the subject-matter of the restriction is still $\boldsymbol{s}$ and that the warm weather in London will at least be relevant to the truth of the restriction.

3 Common Content

I turn to some other concepts that can also be accommodated within the present framework. One of these is the concept of common content. Suppose one person says that Obama is a Muslim President and another says that he is a Christian President. Then one thing they have both said - and, indeed, the common content of what they have both said - is that he is a President. On the other hand, if one person says that Obama is Muslim and another says that he is Christian, then it is no part of the common content of what they both said that he is Muslim or Christian. Indeed, if that were so then his being Muslim or Christian would be part of the content of his being Christian. Thus we cannot in general take the common content of two propositions to be their disjunction.

How then is common content to be defined? Just as in the case of conjunction, we can provide an extrinsic characterization, in terms of the role we wish the relevant kind of content to play, and also an intrinsic characterization, in terms of its constitutive states. The role is clear: the common content of two propositions should be the greatest of their common parts. It should, in other words, be the greatest lower bound of the two propositions with respect to the relation of partial content; and so the notion of common content will be 'vertical dual' to the notion of conjunction.

We may also provide an intrinsic characterization of common content within our framework by taking the common content $P \nabla Q$ of two propositions $P = \{p_1, p_2, ...\}$ and $Q = \{q_1, q_2, ...\}$ to be the proposition $\{p_1 \sqcap \boldsymbol{q}, p_2 \sqcap \boldsymbol{q}, ..., q_1 \sqcap \boldsymbol{p}, q_2 \sqcap \boldsymbol{p}, ...\}$. In effect, each verifier in P and Q is restricted to the common subject-matter of P and of Q. Thus when P is the proposition that Obama is a Muslim President and Q is the proposition that Obama is a Christian President, the common subject-matter will be his being President and the restriction of the verifiers of each proposition to the common subject-matter will be the proposition that he is a President.

We would expect there to be a relationship between the common content $P \nabla Q$ of two propositions and their common subject-matter; and this is indeed so. When $\boldsymbol{p}$ is the subject-matter of P and $\boldsymbol{q}$ of Q, then $\boldsymbol{p} \sqcap \boldsymbol{q}$, the common subject-matter of $\boldsymbol{p}$ and $\boldsymbol{q}$, will be the subject-matter of the common content $P \nabla Q$; and there will, in general, be pleasing connections of this sort between the subject-matter of propositional constructs and the subject-matter of the propositions from which they are constructed.

We may in a similar way define the common disjunctive content of two propositions (which is a 'horizontal dual' to the notion of common conjunctive content). Under the external characterization, the common disjunctive content of two propositions may be taken to be the weakest proposition to entail them both while, under the intrinsic characterization, it may be taken to be the intersection of the two propositions, i.e. the set of verifiers that belongs to both of them.

We may also define the common conjunctive content of two *bilateral* propositions in the obvious way. When the propositions are $\boldsymbol{P} = (P, P')$ and $\boldsymbol{Q} = (Q, Q')$, we may take the common conjunctive content $\boldsymbol{R}$ of $\boldsymbol{P}$ and $\boldsymbol{Q}$ to be the proposition (R, R'), where R is the common conjunctive content of P and Q and R' is the common disjunctive content of P' and Q'; and similarly, though in reverse, for the common disjunctive content.

However, it should be pointed out that the common content, as so defined, may not be very well-behaved. Suppose we take the common conjunctive content of the propositions that a given object is red and that the object is blue. The states of the object being red and of its being green plausibly have the state of the object's existing as their common part; and so we may take the common content of the two propositions, on the positive side, to be that the object exists. On the negative side, the state of the object's being green will plausibly be a common disjunctive part of the propositions that the object is not red and that the object is not blue. Thus the common content will be verified by the existence of the object and falsified by the object's being green, even though the existence of the object is compatible with its being green. One possible solution to this problem, given that our interest is in the common positive content of two propositions, is to determine the falsifiers r' of the common content $\boldsymbol{R}$ of $\boldsymbol{P}$ and $\boldsymbol{Q}$ on the basis of its verifiers R - as its exclusionary negation $\sim R$, for example - rather than on the basis of the respective falsifiers p' and q' of $\boldsymbol{P}$ and $\boldsymbol{Q}$.

4 Differentiated Content

For certain purposes, it is convenient to be able to divide a state s into two parts s_1, s_2 - which we may dub the *pre*-state or -condition and the *post*-state or -condition. Consider, for example, a verifier s for the statement that Hans came too. This will have two parts, the state s_1 of the other people coming and the state s_2 of Hans coming, with the first providing a logical precondition for the second. Or consider a verifier s for the proposition that Hans had one beer and then another. This will again have two parts, the state s_1 of his having the one beer and the state s_2 of his having the other beer, with the first being a temporal pre-condition for the second.

We might call states of this sort *differentiated* and represent them as ordered pairs (s_1, s_2) of undifferentiated states. A natural relation of part-whole on differentiated states can be defined in terms of the corresponding relations of part-whole on the undifferentiated states, with the state (s_1, s_2) being *part of* the state (s_1', s_2') if s_1 is a part of s_1' and s_2 a part of s_2'. A differentiated proposition or content can then be taken to be a set of differentiated states; and regular propositions, subject-matter, common content etc. can be defined in the usual way.

One application of differentiated states and content is to subject-matter. A subject-matter might also be differentiated - into an observational and a theoretical part, for example, or into a nominalistic and platonic part. Given a differentiated subject matter $(\boldsymbol{s}, \boldsymbol{s}')$, we might then define the *restriction of* an undifferentiated proposition $P = \{p_1, p_2, ...\}$ *to* the differentiated subject-matter $(\boldsymbol{s}, \boldsymbol{s}')$ to be the differentiated content $\{(p_1 \sqcap \boldsymbol{s}, p_1 \sqcap \boldsymbol{s}'), (p_2 \sqcap \boldsymbol{s}, p_2 \sqcap \boldsymbol{s}'), ...\}$. Thus each verifier of the proposition is divided into two parts, one concerning the subject matter $\boldsymbol{s}$ and the other concerning the subject-matter $\boldsymbol{s}'$.

Another significant application is to the concept of *coordinated* conjunction. Given two propositions $P = \{p_1, p_2, ...\}$ and $Q = \{q_1, q_2, ...\}$, their conjunction will be the proposition $\{p_1 \sqcup q_1, p_1 \sqcup q_2, ..., p_2 \sqcup q_1, p_2 \sqcup q_2, ..., ...\}$, with each of the p_i fused with each of the q_j. But with coordinated conjunction, all that is required is that each of the p_i be fused with at least one of the q_j and that each of the q_j be fused with at least one of the p_i. Thus, in the case above, $\{p_1 \sqcup q_1, p_2 \sqcup q_2, p_3 \sqcup q_3, ...\}$ would be one of the coordinated conjunctions of P and Q, $\{p_1 \sqcup q_1, p_1 \sqcup q_2, ..., p_2 \sqcup q_2, p_3 \sqcup q_3, ...\}$ would be another, and so on.

To specify a coordinated conjunction we must know which verifiers of the one proposition are to be coordinated with which verifiers of the other; and this may be done by means of a differentiated content. Thus the differentiated content $\{(p_1, q_1), (p_2, q_2), (p_3, q_3), ...\}$ will result in the coordinated conjunction $\{p_1 \sqcup q_1, p_2 \sqcup q_2, p_3 \sqcup q_3, ...\}$. We can think of coordinated conjunction as an operation that 'flattens out' the effect of differentiation, with each differentiated state (p_i, q_i) replaced by the corresponding integrated state $p_i \sqcup q_i$.

From a mathematical point of view, the present constructions are very natural. For differentiated states and contents are what we obtain by taking the product of two state spaces and coordinated conjunctions are what we obtain by 'projecting' a differentiated content down into an undifferentiated space. Indeed, there are a number of familiar mathematical constructions on state spaces (such as the restriction to a subspace or the factoring into a congruent space) which directly relate to a range of different applications.

5 Logical Remainder

One content can be subtracted from another. From the proposition that Obama is a Christian President, for example, we can subtract the content that he is Christian to obtain the proposition that he is a President. We cannot take the result of subtracting Q from P to be the proposition $Q \supset P$, i.e. $\neg Q \vee P$, for the result should be part of P and we do not, in general want $\neg Q \vee P$ to be part of P.

So how should logical subtraction be defined?[5]

Suppose that P is the proposition upon which the subtraction is being performed, that Q is the proposition we wish to subtract from P (and which we assume to be part of P) and that R is the proposition that results from the subtraction. There are then two requirements which may plausibly be imposed upon P, Q and R:

Summation Q and R should sum to P;

Separation R should be separate from Q.

But this leaves open what we mean by 'sum' and 'separate'.

The most natural choice for 'sum' is that Q and R should sum to P in the sense of conjunction: P should be identical (or equivalent) to the conjunction $Q \wedge R$. The most natural choice for 'separate' is that Q and R should be separate or 'disjoint' in subject-matter; $\boldsymbol{q}$ should have no non-null state in common with $\boldsymbol{r}$. Unfortunately, it is very hard to satisfy these two requirements together, as so interpreted.

One difficulty is ontological. At the level of the states themselves, there may be nothing we can sensibly identify as the result of subtracting one state from another. To take an example of Wittgenstein's, it is not clear what is left from my raising my hand when I subtract that my hand rose. So if I take p to be the state of my raising my hand and q to be the state of my hand rising, then it is not clear that there is any proposition $R = \{r_1, r_2, ...\}$ whose conjunction $\{r_1 \sqcup q, r_2 \sqcup q, ...\}$ with $Q = \{q\}$ is the proposition $P = \{p\}$ and whose subject-matter $\boldsymbol{r} = r_1 \sqcup r_2 \sqcup ...$ is disjoint from the subject-matter $\boldsymbol{q}$ of Q.

Another, less commonly appreciated, difficulty is logical. It concerns subtraction from logically complex propositions and arises even when subtraction on the underlying states is well-defined. For consider the proposition P expressed by the formula $(\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{r} \wedge \mathrm{s})$, where p, q, r and s express independent propositions; and suppose we wish to subtract the proposition Q expressed by the formula $\mathrm{p} \vee \mathrm{r}$. What then is the remainder R? One might have thought that it was the proposition expressed by $\mathrm{q} \vee \mathrm{s}$. But the conjunction $(\mathrm{p} \vee \mathrm{r}) \wedge (\mathrm{q} \vee \mathrm{s})$ does not imply $(\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{r} \wedge \mathrm{s})$, since $(\mathrm{p} \vee \mathrm{r}) \wedge (\mathrm{q} \vee \mathrm{s})$ is true when just p and s are true while $(\mathrm{p} \wedge \mathrm{q}) \vee (\mathrm{r} \wedge \mathrm{s})$ is not. If

[5] A question raised in [10] and subsequently discussed by [8, 9, 11] and [12]. I have not attempted to compare Humberstone's and Yablo's accounts with my own, though the simplicity and theoretical naturalness of my own account is perhaps a strong point in its favor.

we insist upon summation through conjunction, then it looks as if R must be a proposition expressed by a formula R containing both p and r; and so R will not be disjoint in subject-matter from Q.[6]

We shall not attempt to evade the ontological difficulty but shall simply allow that, when ontological circumstances are unpropitious, the logical remainder of two propositions may not exist. The logical difficulty, by contrast, may to some extent be removed by relaxing Summation. Instead of insisting that P be the conjunction of Q and R, we need only require that it be a coordinated conjunction of Q and R. Thus the role of the remainder R is not to conjoin but to embellish; the verifiers of Q, in turn, are expanded via the verifiers of R to yield the verifiers of P. The proposition expressed by q $\vee$ s will then be a remainder in the example above since, under the coordination of p with q and of r with s, p $\vee$ r and q $\vee$ s will sum to (p $\wedge$ q) $\vee$ (r $\wedge$ s).

The present combination of requirements - of coordinated conjunction, on the one hand, and disjoint subject-matter, on the other - provides a very satisfying account of logical subtraction. If the remainder R of subtracting Q from P exists, then it will be unique and its subject-matter $\boldsymbol{r}$ will be the result $\boldsymbol{p}$ - $\boldsymbol{q}$ of subtracting the subject-matter $\boldsymbol{q}$ of Q from the subject-matter $\boldsymbol{p}$ of P. There will also be a simple internal characterization of the remainder R. For where P is the proposition $\{p_1, p_2, ...\}$ and Q is a proposition whose subject-matter is $\boldsymbol{q}$, the logical remainder $P - Q$ will be the proposition $\{p_1 - \boldsymbol{q}, p_2 - \boldsymbol{q}, ...\}$, obtained by subtracting the subject-matter of Q from each verifier of P.

However, there will still be cases in which this account will be incapable of yielding satisfactory results. For taking p and q to express independent propositions, the result R of subtracting the proposition Q expressed by p $\vee$ q from the proposition P expressed by p $\wedge$ q will be the completely trivial proposition, whose sole verifier is the null state; and, in this case, the coordinated conjunction of R with Q will simply give back Q.

We may also define a somewhat weaker notion of logical remainder by insisting, not on separation in *subject-matter*, but on separation in *content*. Say that a proposition is *trivial* if the null state $\square$ is one of its verifiers. Then propositions Q and R are *separate in content* if no non-trivial proposition is part of them both. Weakening the requirement on separation enables us to obtain a broader class of logical remainders, which will generally exist even though the stricter requirement cannot be met.

6 Ground

Perhaps one of the most surprising aspects of the present theory is its connection with the 'worldly' conception of ground adumbrated in [2]. The distinction between the worldly and conceptual conception of ground may be roughly explained as follows. A statement represents the world as being a certain way. We may therefore distinguish between the way it represents the world as being and how it represents the world as

[6]Suppose that R did not contain p, for example. Then the truth of Q $\wedge$ R could not turn on the truth-value of p when r and q were true.

being that way. The worldly content of the statement is just a matter of the way it represents the world, while the conceptual content is also a matter of how it represents that content. The worldly conception of ground is one that is blind to anything other than factual content, while the conceptual conception of ground is one that also takes into account the representation of the factual content. Thus the worldly conception will presumably not distinguish between P and P $\wedge$ P, since these two statements represent the world as being the same and so, just as it would be incorrect to say that P grounds P, it would be incorrect, under the worldly conception of ground, to say P grounds P $\wedge$ P. On the conceptual conception of ground, by contrast, it will be perfectly acceptable to say that P grounds P $\wedge$ P since the one representation will be true in virtue of the other.

There are, of course, a number of different ways in which one might attempt to draw the distinction between worldly and conceptual content. When it comes to statements of truth-functional logic, [2] opts for a view in which two statements are taken to have the same worldly content just in case they are analytically equivalent in the sense of Angell [1]. Now it turns out that Angell's logic of analytic equivalence is exactly the logic we obtain if we take the sentence-letters of our language to stand for regular verifiable propositions and interpret the Boolean operators on formulas by means of the corresponding Boolean operations on regular propositions [5]. This therefore suggests that we should take the worldly content of a statement to be the associated regular proposition. In effect, the worldly content of a statement is given, in the most natural way possible, by the set of states or facts that make it true (along with the set of states or facts that make it false).

We still need to define the notion of ground on such propositions and here we may follow the lead of [3], which attempts to provide a semantics for the pure logic of ground. Thus (focusing on the unilateral case and giving the definitions purely at the level of content), P weakly grounds Q if P entails Q and P_1, P_2, ... weakly grounds Q if their conjunction weakly grounds Q. P_1, P_2, ... strictly grounds Q (i.e. grounds Q in the customary sense of the term) if (i) P_1, P_2, ... weakly grounds Q and (ii) Q along with any other propositions does not weakly ground any of the propositions P_1, P_2, ... It turns out that, within the domain of verifiable regular propositions, the second condition (ii) can be equivalently formulated as the condition that the subject matter $\boldsymbol{q}$ of the grounded proposition Q should properly contain the subject matters $\boldsymbol{p}_1, \boldsymbol{p}_2$, ... of each of the grounding propositions P_1, P_2, ... Thus the explanatory progress characteristic of strict ground is seen to consist in the enlargement of subject-matter as we pass from the propositions that do the grounding to those that are grounded.

The resulting definition of ground is remarkable in a number of ways. First, the definition is given in purely logical terms (via the operations of conjunction and disjunction). Thus, from the present perspective, ground is a logical rather than a metaphysical notion. Second, the definition can in fact be given, rather simply, in terms of the two basic notions of consequence - entailment and containment - and so can properly be regarded as a bi-product of the general theory of consequence within the theory of regular propositions. Third, the definition turns out to be in agreement with the definition given by Correia [2010]. We thereby achieve a remarkable consilience between two separate strands of thought, with the present account of ground in terms of the application of the semantics for the pure logic of ground to the theory of regular propositions yielding the very same result as Correia's independently motivated account in terms of Angell's notion of analytic entailment.

I should add that the present account provides a vindication of the notion of *weak* ground, not merely of its intelligibility but also of its significance in the investigation of ground. For we may wish to break down the demonstration that certain propositions ground another into its simplest steps. Consider now the fact that P, Q strictly grounds $(P \wedge Q) \vee Q$. Then we naturally proceed by first showing that P, Q strictly grounds $(P \wedge Q)$ and then showing that $(P \wedge Q)$ grounds $(P \wedge Q) \vee Q$. But this last relationship of ground is weak, since the subject-matters of $(P \wedge Q)$ and $(P \wedge Q) \vee Q$ are the same (or, alternatively, since $(P \wedge Q) \vee Q$, P weakly grounds $P \wedge Q)$. Thus a step of weak ground is required to show that certain given propositions are a (strict) ground for another; and there would appear to be no way in which the appeal to weak ground might be avoided. So even if our initial interest is in strict ground, appeal to weak ground may be required to demonstrate the relationships of strict ground. The case of $(P \wedge Q)$ grounding $(P \wedge Q) \vee Q$ also illustrates how relationships of weak ground are not always cases of ground-theoretic equivalence (as exists, for example, between P and $P \wedge P$); and so we should not assume that the work done by weak ground can always be done by substituting ground-theoretic equivalence in its place.

Appendix: Formal Appendix

Preliminaries

The technical appendix from part I is presupposed; and results from that appendix are prefixed with a 'I', as with 'lemma I.1'.

We define a product space in the usual way. Given two spaces $\boldsymbol{S} = (S, \sqsubseteq)$ and $\boldsymbol{S}' = (S', \sqsubseteq')$, their *product space* $\boldsymbol{S} \times \boldsymbol{S}'$ will be $(S \times S', \sqsubseteq^*)$, where:

$(s, t) \sqsubseteq^* (s', t')$ iff $s \sqsubseteq s'$ and $t \sqsubseteq t'$.

When $\boldsymbol{S}$ and $\boldsymbol{S}$' are modalized spaces $(S, S^{\Diamond}, \sqsubseteq)$ and $(S', S'^{\Diamond}, \sqsubseteq')$, we may take their product $\boldsymbol{S} \times \boldsymbol{S}'$ to be $(S \times S', (S \times S')^{\Diamond}, \sqsubseteq^*)$, with $\sqsubseteq^*$ as before and with $(S \times S')^{\Diamond} = (S^{\Diamond} \times S'^{\Diamond})$ (although more restrictive definitions of $(S \times S')^{\Diamond}$ might also be given).

We readily show:

Lemma 1 *If* $\boldsymbol{S}$ *and* $\boldsymbol{S}'$ *are modalized or unmodalized state spaces then so is* $\boldsymbol{S} \times \boldsymbol{S}'$*, with* $(s_1, t_1) \sqcup^* (s_2, t_2) \sqcup^* \ ... = (s_1 \sqcup s_2 \sqcup \ ..., t_1 \sqcup t_2 \sqcup ...\}$ *for* $s_1, s_2, ... \in S$ *and* $t_1, t_2, ... \in T$.

We say that r *is a remainder of* t given $s \sqsubseteq t$ if (i) $r \sqcup s = t$ and (ii) r and s are disjoint; and the space $\boldsymbol{S}$ is said to be *remaindered* if it is distributive and if for any states $s, t \in S$ with $s \sqsubseteq t$, there is a remainder of t given s. In a remaindered space, the remainder r of t given $s \sqsubseteq t$ is unique; and we denote it by $t - s$. We may extend the notion of remainder to arbitrary t and s by taking $t - s = t - (t \sqcap s)$.

We state without proof the following facts about remainders within a remaindered space, to which implicit appeal will be made in the proofs to follow:

(i) $(t - u) \sqcup u = t$ for $u \sqsubseteq t$
(ii) $t - u$ and u are disjoint
(iii) if $s \sqsubseteq t$ and s is disjoint from $u \sqsubseteq t$ then $s \sqsubseteq t - u$
(iv) if $t \sqsubseteq t'$ and $s' \sqsubseteq s$ then $t - s \sqsubseteq t' - s'$
(v) $(s \sqcup t) - u = (s - u) \sqcup (t - u)$

Common Conjunctive Part

Given regular verifiable propositions P_1, P_2, ..., we take their *mereological intersection* (*common conjunctive part*) $P = P_1 \nabla P_2 \nabla$ ... to be $\{q$: $q = p \sqcap \bigsqcap \boldsymbol{p}_i$ for some $p \in P_1 \cup P_2 \cup ...\}$ when there are some P_1, P_2, ... and to be $F_\blacksquare = \{\blacksquare\}$ otherwise. For each i, $\boldsymbol{p}_i$ is the subject-matter of P_i and so $\bigsqcap \boldsymbol{p}_i$ is the common subject matter of P_1, P_2, ... Thus $P_1 \nabla P_2 \nabla$ ... is obtained by restricting the verifiers in P_1, P_2, ... to their common subject matter.

We have the following external characterization of common conjunctive part:

Theorem 2 *Suppose that P_1, P_2, ... are regular verifiable propositions and that $P = P_1 \nabla P_2 \nabla$ .... Then:*

(i) $\boldsymbol{p} = \bigsqcap \boldsymbol{p}_i$
(ii) *P is the greatest regular verifiable proposition to be contained in each of P_1, P_2, ...*
(iii) *P is the conjunction of all the regular verifiable propositions to be contained in each of P_1, P_2, ...*

Proof (i) When there are no P_1, P_2, ..., $P = F_\blacksquare$ and $\bigsqcap \boldsymbol{p}_i = \blacksquare = \boldsymbol{p}$. So suppose there are some P_1, P_2, ... Then $\boldsymbol{p} = \bigsqcup\{p \sqcap \bigsqcap \boldsymbol{p}_i$: $p \in P_1 \cup P_2 \cup ...\} = (\boldsymbol{p}_1 \sqcap \bigsqcap \boldsymbol{p}_i) \sqcup (\boldsymbol{p}_2 \sqcap \bigsqcap \boldsymbol{p}_i) \sqcup$ ..., given that each $\boldsymbol{p}_i$ is the maximal member of P_i. But $(\boldsymbol{p}_1 \sqcap \bigsqcap \boldsymbol{p}_i) \sqcup (\boldsymbol{p}_2 \sqcap \bigsqcap \boldsymbol{p}_i) \sqcup ... = \bigsqcap \boldsymbol{p}_i \sqcap \bigsqcap \boldsymbol{p}_i \sqcup ... = \bigsqcap \boldsymbol{p}_i$.

(ii) The result follows from lemma I.2 when there are no P_i, since $\boldsymbol{p} = F_\blacksquare$ is the greatest regular verifiable proposition. So suppose that there are some P_i and that p, say, is a member of P_1. Then $p \sqcap \bigsqcap \boldsymbol{p}_i \in P$ and so P is verifiable. We establish the following further facts in completing the demonstration of (ii):

(1) P is regular. (a) $\boldsymbol{p} = \bigsqcap \boldsymbol{p}_i$ by (i) and $\bigsqcap \boldsymbol{p}_i = (\boldsymbol{p}_1 \sqcap \bigsqcap \boldsymbol{p}_i) \in P$. (b) Suppose $q \in P$ and $q \sqsubseteq q' \sqsubseteq \bigsqcap \boldsymbol{p}_i$. Then q is of the form $p \sqcap \bigsqcap \boldsymbol{p}_i$ for some $p \in P_i$. Since $q' \sqsubseteq \bigsqcap \boldsymbol{p}_i \sqsubseteq \boldsymbol{p}_i \in P_i$ and P_i is a regular proposition, $p \sqcup q' \in P_i$. But then $(p \sqcup q') \sqcap \bigsqcap \boldsymbol{p}_i = (p \sqcap \bigsqcap \boldsymbol{p}_i) \sqcup (q' \sqcup \bigsqcap \boldsymbol{p}_i) = q \sqcup q' = q' \in P$.

(2) P is contained in each of P_1, P_2, ... Take $p \in P$. Then p is of the form $P' \sqcap \bigsqcap \boldsymbol{p}_i$ with $P' \in P_i$ for some i. But then $p = p' \sqcap \bigsqcap \boldsymbol{p}_i \sqsubseteq \boldsymbol{p}_i \in P_i$. Now suppose $p \in P_i$. Then $p \sqsupseteq p \sqcap \bigsqcap \boldsymbol{p}_i \in P$.

(3) P is the greatest proposition to be contained in each of P_1, P_2, ... Take a proposition R that is contained in each P_i. We wish to show $R \leqslant P$.

Suppose first that $r \in R$. Since $R \leqslant P_i$ for each i, $r \sqsubseteq \boldsymbol{p}_i$ for each i, and so $r \sqsubseteq \bigsqcap \boldsymbol{p}_i \in P$. Now suppose $p \in P$. Then p is of the form $P' \sqcap \bigsqcap \boldsymbol{p}_i$ with $P' \in P_i$ for some i. So $P' \sqsupseteq r$ for some $r \in R$ given that $R \leqslant P_i$. Also each $\boldsymbol{p}_i \sqsupseteq r$, given that R is contained in each P_i, and hence $\bigsqcap \boldsymbol{p}_i \sqsupseteq r_i$. But then $p = p' \sqcap \bigsqcap \boldsymbol{p}_i \sqsupseteq r$.

(iii) From (ii) by theorem I.12. □

Note that, in the special case of two propositions P and Q, we may take their common content $P \triangledown Q$ to be $\{p \sqcap \boldsymbol{q}: p \in P\} \cup \{q \sqcap \boldsymbol{p}: q \in Q\}$. The characterization of the common content of two propositions also simplifies when one of them is definite:

Corollary 3 *If P and Q are regular verifiable propositions and one of them is definite, then $P \triangledown Q = \{p \sqcap q: p \in P$ and $q \in Q\}$.*

Proof Without loss of generality, assume $Q = \{q_0\}$. Then:

$P \triangledown Q = \{p \sqcap \boldsymbol{q}: p \in P\} \cup \{q \sqcap \boldsymbol{p}: q \in Q\}$

$= \{p \sqcap q_0: p \in P\} \cup \{q_0 \sqcap \boldsymbol{p}\}$

$= \{p \sqcap q: p \in P$ and $q \in Q\}$ since $\boldsymbol{p} \in P$ and $Q = \{q_0\}$. □

A greatest common part of P and Q may not exist when P and Q are only required to be semi-regular. For let $P = \{pq, rs\}$ and $Q = \{pr, qs\}$ (in the canonical space). Then P and Q are both semi-regular propositions, {p, r} and {q, s} are both maximal common parts of P and Q, and yet neither is a part of the other. When we turn, on the other hand, to the regular closures P^*_* and Q^*_* of P and Q, their greatest common part will be $\{pq, rs, pr, qs\}^*_*$.

The common part of one or more regular propositions is related to their disjunction, with the common part of the propositions being identical to the common part of their disjunction and their common subject-matter:

Lemma 4 *Where P_1, P_2, ... are one or more regular verifiable propositions, $P_1 \triangledown P_2 \triangledown ... = (P_1 \vee P_2 \vee ...) \triangledown \{\bigsqcap \boldsymbol{p}_i\}$.*

Proof Suppose $q \in P_1 \triangledown P_2 \triangledown ....$ Then q is of the form $p \sqcap \bigsqcap \boldsymbol{p}_i$ for $p \in P_1 \cup P_2 \cup ....$ But $\bigsqcap \boldsymbol{p}_i = \bigsqcup\{\bigsqcap \boldsymbol{p}_i\}$ and so $q = p \sqcap \bigsqcap \boldsymbol{p}_i \in (P_1 \vee P_2 \vee ...) \{\bigsqcap \boldsymbol{p}_i\}$. Now suppose $q \in (P_1 \vee P_2 \vee ...) \triangledown \{\bigsqcap \boldsymbol{p}_i\}$. Then q is of the form $p \sqcap \bigsqcap \boldsymbol{p}_i$ where, for some i and $p_i \in P_i$, $p_i \sqsubseteq p \sqsubseteq \boldsymbol{p}_1 \sqcup \boldsymbol{p}_2 \sqcup ....$ But q is also identical to $(p \sqcap \boldsymbol{p}_i) \sqcap \bigsqcap \boldsymbol{p}_i$, where $p_i \sqsubseteq (p \sqcap \boldsymbol{p}_i) \sqsubseteq \boldsymbol{p}_i$ and hence $(p \sqcap \boldsymbol{p}_i) \in P_i$; and so $q \in P_1 \triangledown P_2 \triangledown ....$ □

Common Disjunctive Part

We also have a 'vertical' dual to disjunction. Given the propositions P_1, P_2, ..., we let their *logical intersection* (*common disjunctive part*) P be $P_1 \triangledown P_2 \triangledown ... = P_1 \cap P_2 \cap ....$ Note that there is no guarantee that P will be non-empty when P_1, P_2, ...

are non-empty and that there will, in general, be no way to determine the subject-matter of P from the subject matter of P_1, P_2, ... Thus within the canonical space, the subject-matter of {p, q} and {pq} will be the same, viz. pq, while the common subject-matters {p, q, pq} △ {p} = {p} and {pq} △ {p} = ∅ will be different.

Theorem 5 *Suppose that* P_1, P_2, ... *are regular propositions and that* $P = P_1 \triangle P_2 \triangle \ldots$ *. Then:*

(i) *P is the weakest regular proposition to entail each of P_1, P_2, ...;*
(ii) *P is the disjunction of all the regular propositions to entail each of P_1, P_2, ...*

Proof (i) This is clearly true if P is empty and so we may suppose P is non-empty. We establish the following facts in turn:

(1) $\boldsymbol{p} \in P$. Each $p \in P$ belongs to each P_i; so $\boldsymbol{p} = \bigsqcup P$ belongs to each P_i, given that the P_i are regular, and so $\boldsymbol{p} \in P$.
(2) P is convex. Suppose $p \sqsubseteq q \sqsubseteq r$ with $p, r \in P$. Then p, r belongs to each P_i; so q belongs to each P_i, given that the P_i are regular; and so $q \in P$.
(3) P is regular. From (1) and (2).
(4) P is the weakest regular proposition to entail each P_i. Evident from the definition of P.

(ii) From (i) by theorem I.14.

□

Common Conjunctive and Disjunctive Part - the Bilateral Case

Given regular verifiable propositions $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ..., we let their *mereological intersection* (*common conjunctive part*) $\boldsymbol{P} = \boldsymbol{P}_1 \triangledown \boldsymbol{P}_2 \triangledown \ldots$ be $(P_1 \triangledown P_2 \triangledown \ldots, P_1' \triangle P_2' \triangle \ldots)$; and given regular falsifiable propositions $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ..., we let their *logical intersection* (*common disjunctive part*) $\boldsymbol{P} = \boldsymbol{P}_1 \triangle \boldsymbol{P}_2 \triangle \ldots$ be $(P_1 \triangle P_2 \triangle \ldots, P_1' \triangledown P_2' \triangledown \ldots)$. These definitions are in perfect analogy to the definitions of conjunction and disjunction in the bilateral case.

Theorems 2 and 5 may be extended to bilateral propositions in the obvious way. However, there is a peculiar difficulty in the present case. Within a Boolean domain, we can be sure that the conjunction $\boldsymbol{P}^c$ of all the regular verifiable propositions to be contained in each of the given verifiable propositions $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... will exist and that the disjunction $\boldsymbol{P}^d$ of all the regular propositions to entail each of $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... will exist; and we can also be sure that if $\boldsymbol{P}_1 \triangledown \boldsymbol{P}_2 \triangledown \ldots$ belongs to the propositional domain then it will be identical to the conjunction $\boldsymbol{P}^c$ and that if $\boldsymbol{P}_1 \triangle \boldsymbol{P}_2 \triangle \ldots$ belongs to the propositional domain then it will be identical to the disjunction $\boldsymbol{P}^d$. But we can have no general assurance either that $\boldsymbol{P}_1 \triangledown \boldsymbol{P}_2 \triangledown \ldots$ or $\boldsymbol{P}_1 \triangle \boldsymbol{P}_2 \triangle \ldots$ will exist in suitably constrained domains or that they will inherit desirable properties of the component propositions $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ..., such as bivalence, should they exist (a case of this sort is discussed in the introduction).

It would be of interest to determine the conditions for the common conjunctive or disjunctive part of given bilateral propositions to be 'well-behaved', under different determinations of what it is for a proposition to be well-behaved. Another line of solution is to let the falsifiers of the common conjunctive or disjunctive part 'fall where they may'. Thus given the verifiable propositions $\boldsymbol{P}_1, \boldsymbol{P}_2, ...$, we may let their common conjunctive content be $((P_1 \nabla P_2 \nabla ...), \sim(P_1 \nabla P_2 \nabla ...))$, where $\sim(P_1 \nabla P_2 \nabla ...)$ is the exclusionary negation of $(P_1 \nabla P_2 \nabla ...)$ as defined in part I; and similarly for the common disjunctive part. The common conjunctive or disjunctive part is defined, in effect, in terms of the positive content of the given propositions; and the negative content is ignored. But even in this case, it will be necessary to allow there to be trivial propositions distinct from $T_{\square}$, since in many cases the common unilateral content will be such a proposition, as with the common conjunctive content {p, □} of {p, q} and {p}.

Differentiated Content

Differentiated spaces will be helpful in developing the theory of logical remainder and subject-matter.

Given an ordinary (*undifferentiated*) space $\boldsymbol{S}$, we take the corresponding *differentiated* space to be the product space $\boldsymbol{S} \times \boldsymbol{S}$, as previously defined. Intuitively, we think of each state s from $\boldsymbol{S}$ as being differentiated into two parts s_1, s_2, with $s = s_1 \sqcup s_2$, thereby giving us a differentiated state (s_1, s_2) within $\boldsymbol{S} \times \boldsymbol{S}$.

Given a differentiated state $\pi = (s, t) \in \mathrm{S} \times S$, there are three states from S that may be associated with it: the *initial state* $\pi_1 = s$, the *additional state* $\pi_2 = t$, and the *total state* $\pi_{1,2} = s \sqcup t$. Similarly, given a differentiated content $\Pi \subseteq S \times S$ from $\boldsymbol{S} \times \boldsymbol{S}$, there are three associated contents: the *initial content* $\Pi_1 = \{\pi_1 : \pi \in \Pi\}$ (= $\{s \in S$: for some $t \in S, (s, t) \in \Pi\}$), the *additional content* $\Pi_2 = \{\pi_2 : \pi \in \Pi\}$ (= $\{t \in S$: for some $s \in S, (s, t) \in \Pi\}$); and the *total content* $\Pi_{1,2} = \{\pi_{1,2} : \pi \in \Pi\}$ (= $\{s \sqcup t : (s, t) \in \Pi\}$).

Lemma 6 *If* Π *is a regular content in the differentiated space* $\boldsymbol{S} \times \boldsymbol{S}$*, then* Π_1, Π_2 *and* $\Pi_{1,2}$ *are regular contents in the undifferentiated space* $\boldsymbol{S}$.

Proof Suppose Π is a regular content. The results are evident when $\Pi = \varnothing$ and so let us suppose Π is non-empty.

(1) Π_1 is regular. For suppose $\Pi_1 = \{s_1, s_2, ...\}$. Then for each i, there is a state t_i for which $(s_i, t_i) \in \Pi$. (a) $(s_1, t_1) \sqcup (s_2, t_2) \sqcup ... = (s_1 \sqcup s_2 \sqcup ..., t_1 \sqcup t_2 \sqcup ...) \in \Pi$, given that Π is regular; and so $\bigsqcup \Pi_1 = s_1 \sqcup s_2 \sqcup ... \in \Pi_1$. (b) Now suppose $s \sqsubseteq s' \sqsubseteq s^+$, with $s, s^+ \in \Pi_1$. Then for some $t, t^+ \in S$, (s, t), $(s^+, t^+) \in \Pi$. Since Π is regular, $(s \sqcup s^+, t \sqcup t^+) = (s^+, t \sqcup t^+) \in \Pi$ and, given that $(s, t) \sqsubseteq (s', t) \sqsubseteq (s^+, t \sqcup t^+)$, $(s', t) \in \Pi$ and hence $s' \in \Pi_1$.

(2) Π_2 is regular. Similar to (1).

(3) $\Pi_{1,2}$ is regular. (a) Let $\Pi_1 = \{s_1, s_2, ...\}$ and $\Pi_2 = \{t_1, t_2, ...\}$. Then $\bigsqcup \Pi_{1,2} = \bigsqcup \Pi_1 \sqcup \bigsqcup \Pi_2$. But $\bigsqcup \Pi = (\bigsqcup \Pi_1, \bigsqcup \Pi_2) \in \Pi$ and so $\bigsqcup \Pi_{1,2} = \bigsqcup \Pi_1 \sqcup \bigsqcup \Pi_2 \in \Pi_{1,2}$. (b) Suppose $s \sqcup t \sqsubseteq u \sqsubseteq s' \sqcup t'$, with (s, t), $(s', t') \in \Pi$.

Since Π is regular, $(s \sqcup s', t \sqcup t') \in \Pi$. So without loss of generality, we may suppose $s \sqsubseteq s'$ and $t \sqsubseteq t'$. But $u = u \sqcup (s' \sqcup t') = (u \sqcup s') \sqcup (u \sqcup t')$, $s \sqsubseteq (u \sqcup s') \sqsubseteq s'$ and $t \sqsubseteq (u \sqcup t') \sqsubseteq t'$; and so $(u \sqcup s', u \sqcup t') \in \Pi$, given that Π is regular, from which it follows that $u = (u \sqcup s') \sqcup (u \sqcup t') \in \Pi_{1,2}$. □

Coordinated Conjunction

For Π a differentiated content and R, Q, P undifferentiated verifiable contents: we say P is the Π-*conjunction of* R *and* Q - in symbols, $R \wedge_\Pi Q = P$ - if $\Pi_1 = R$, $\Pi_2 = Q$ and $\Pi_{1,2} = P$; and we say P is *a coordinated conjunction of* R *and* Q- or, with an abuse of notation, $R \wedge_\bullet Q = P$ - if $R \wedge_\Pi Q = P$ for some regular differentiated content Π. Note that $R \wedge_\Pi Q$ is not taken to be defined unless R and Q are verifiable, $\Pi_1 = R$ and $\Pi_2 = Q$.

In a coordinated conjunction, the 'conjuncts' are coordinated with one another via a differentiated content, with the 'disjunct' r in R fused with the disjunct q in Q just in case (r, q) is a state within the coordinating content Π. The ordinary conjunction $R \wedge Q$ is the special case of a coordinated conjunction $R \wedge_\Pi Q$ in which $\Pi = R \times Q$.

The following results follow straightforwardly from the definitions:

Lemma 7 *(i)* $R \leqslant_c R \wedge_\Pi Q$ *and* $Q \leqslant_c R \wedge_\Pi Q$*;*
(ii) $\Pi \subseteq \Pi'$ *implies* $R \wedge_\Pi Q \leqslant_d R \wedge_{\Pi'} Q$ *for differentiated contents* $\Pi \subseteq \Pi'$*, and* $\Pi \subseteq \Pi'$ *implies* $R \wedge_{\Pi'} Q \leqslant_c R \wedge_\Pi Q$ *for regular differentiated contents* $\Pi \subseteq \Pi'$*;*
(iii) $R \wedge_\Pi Q \leqslant_d R \wedge Q$ *and* $R \wedge Q \leqslant_c R \wedge_\Pi Q$ *for regular* Π.

Coordinated conjunction is preserved under regular closure:

Lemma 8 *Suppose* $R \wedge_\Pi Q = P$*. Then* $R^*_* \wedge_{\Pi^*} Q^*_* = P^*_*$*.*

Proof It should be clear that $\bigsqcup \Pi = (\boldsymbol{r}, \boldsymbol{q})$ and so $\Pi^*_* = \{(r', q')\colon (r, q) \in \Pi, r \sqsubseteq r' \sqsubseteq \boldsymbol{r}$ and $q \sqsubseteq q' \sqsubseteq \boldsymbol{q})\}$.

(1) $\Pi^*_{*1} \subseteq R^*_*$ and $\Pi^*_{*2} \subseteq Q^*_*$.
Pf. Suppose $(r', q') \in \Pi^*_*$ (to show $r' \in R^*_*$ and $q' \in Q^*_*$). Then for some $(r, q) \in \Pi$, $r \sqsubseteq r' \sqsubseteq \boldsymbol{r}$ and $q \sqsubseteq q' \sqsubseteq \boldsymbol{q}$. Given that $R \wedge_\Pi Q = P$, $r \in R$, $q \in Q$; $\boldsymbol{r} \in R^*_*$ and $\boldsymbol{q} \in \mathrm{Q}^*_*$; and so, given that $r \sqsubseteq r' \sqsubseteq \boldsymbol{r}$ and $q \sqsubseteq q' \sqsubseteq \boldsymbol{q}$, $r' \in R^*_*$ and $q' \in Q^*_*$.

(2) $R^*_* \subseteq \Pi^*_{*1}$ and $Q^*_* \subseteq \Pi^*_{*2}$.
Pf. Suppose $r' \in R^*_*$ (the case in which $q' \in Q^*_*$ is similar). Then $r \sqsubseteq r' \sqsubseteq \boldsymbol{r}$ for some $r \in R$. So $(r, q) \in \Pi$ for some $q \in Q$. But then $r \sqsubseteq r' \sqsubseteq \boldsymbol{r}$ and $q \sqsubseteq q \sqsubseteq \boldsymbol{q}$; and so $(r', q) \in \Pi^*_*$, from which it follows that $R^*_* \subseteq \Pi^*_{*1}$.

From (1) and (2), we obtain:

(3) $\Pi^*_{*1} = R^*_*$ and $\Pi^*_{*2} = Q^*_*$.

(4) $\Pi^*_{*1,2} \subseteq P^*_*$.

Pf. Take $(r', q') \in \Pi^*_*$, with $(r, q) \in \Pi$, $r \sqsubseteq r' \sqsubseteq \boldsymbol{r}$ and $q \sqsubseteq q' \sqsubseteq \boldsymbol{q}$. Since $r \in R$ and $q \in Q$, $r \sqcup q \in P^*_*$ and, since $\bigsqcup \Pi = (\boldsymbol{r}, \boldsymbol{q})$, $\boldsymbol{r} \sqcup \boldsymbol{q} \in P^*_*$. But $r \sqcup q \sqsubseteq r' \sqcup q' \sqsubseteq \boldsymbol{r} \sqcup \boldsymbol{q}$; and so $r' \sqcup q' \in P^*_*$.

(5) $P^*_* \subseteq \Pi^*_{*1,2}$.

Pf. Suppose $P' \in P^*_*$. Then for some $p \in P$, $p \sqsubseteq p' \sqsubseteq \boldsymbol{p} = \boldsymbol{r} \sqcup \boldsymbol{q}$. But p is of the form $r \sqcup q$ for $(r, q) \in \Pi$. Now $p' = p' \sqcap (\boldsymbol{r} \sqcup \boldsymbol{q}) = (p' \sqcap \boldsymbol{r}) \sqcup (p' \sqcap \boldsymbol{q})$, $r \sqsubseteq (P' \sqcap \boldsymbol{r}) \sqsubseteq \boldsymbol{r}$ and $q \sqsubseteq (p' \sqcap \boldsymbol{q}) \sqsubseteq \boldsymbol{q}$. So $(p' \sqcap \boldsymbol{r}, p' \sqcap \boldsymbol{q}) \in \Pi^*_*$ and, consequently, $p' = (p' \sqcap \boldsymbol{r}) \sqcup (p' \sqcap \boldsymbol{q}) \in \Pi^*_{*1,2}$.

From (4) and (5), we obtain:

(6) $\Pi^*_{*1,2} = P^*_*$.

The required result then follows from (3) and (6). □

Logical Remainder

We deal with two kinds of remainder - first, those required to be *least* with respect to the containment and, second, those required to be *weakest* with respect to entailment.

Suppose P and Q are regular verifiable propositions within a remainder space. We then define $P - Q$ to be $\{p - \boldsymbol{q}: p \in P\}$. Thus $P - Q$ is obtained, so to speak, by subtracting the subject-matter of Q from P.

It should be noted that the value of $R = P - Q$, for given P, depends only upon the subject-matter $\boldsymbol{q}$ of Q and not on its actual constitution. However, when $Q \leqslant_c P$, the identity of P and R will tell us something about the identity of Q. For given $p \in P$, there will be a $q \in Q$ for which $q \sqsubseteq p$. Let $q_p = \bigsqcup\{q \in Q: q \sqsubseteq p\}$. Since Q is regular, $q_p \in Q$ and, clearly, for any $q \in Q$ for which $q \sqsubseteq p$, $q \sqsubseteq q_p$. Thus q_p is the maximal member q of Q for which $q \sqsubseteq p$. But we now see that $p - \boldsymbol{q} = p - q_p$. For $p - \boldsymbol{q} = p - (\boldsymbol{q} \sqcup p)$ by definition. Given $q \in Q$ for which $q \sqsubseteq p$, $q \sqsubseteq (\boldsymbol{q} \sqcup p) \sqsubseteq \boldsymbol{q}$ and so $(\boldsymbol{q} \sqcup p)$ is a maximal member q of Q for which $q \sqsubseteq p$ and hence is identical to q_p. Thus for each $p \in P$, $q_p = (\boldsymbol{q} \sqcup p) \in Q$ and, setting $r_p = p - q_p$, $P - Q = \{r_p: p \in P\}$.

Lemma 9 *Suppose that P and Q are regular verifiable propositions within a remaindered space. Then:*

(i) $R = P - Q$ *is a regular verifiable proposition with* $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{q}$, *and*
(ii) $P - Q = P - (P \nabla Q)$.

Proof (i) Clearly R is verifiable given that P and Q are verifiable. Now $p - \boldsymbol{q} \sqsubseteq \boldsymbol{p} - \boldsymbol{q}$ and, given that $\boldsymbol{p} \in P$, $\boldsymbol{r} = \bigsqcup\{p - \boldsymbol{q}: p \in P\} = \boldsymbol{p} - \boldsymbol{q}$.

Now suppose $r \sqsubseteq r' \sqsubseteq \boldsymbol{p} - \boldsymbol{q}$, with $r \in R$ (to show $r' \in R$). Then r is of the form $p - \boldsymbol{q}$ for $p \in P$. Let $p' = r' \sqcup p$. Then $p' \in P$, since $r' \sqsubseteq \boldsymbol{p}$ and so $p \sqsubseteq p' = r' \sqcup p \sqsubseteq \boldsymbol{p}$. Hence $p' - \boldsymbol{q} \in R$. But $p' - \boldsymbol{q} = (r' \sqcup p) - \boldsymbol{q} = (r' - \boldsymbol{q}) \sqcup (p - \boldsymbol{q}) = r' \sqcup r = r'$.

(ii) $P - Q = \{p - \boldsymbol{q} : p \in P\} = \{p - (\boldsymbol{p} \sqcup \boldsymbol{q}): p \in P\} = P - (P \nabla Q)$ since $\bigsqcup(P \nabla Q) = \boldsymbol{p} \sqcup \boldsymbol{q}$ by theorem 2.

□

Given regular verifiable propositions P and Q, we say that R is a *remainder of* P *from* Q if P is a coordinated conjunction of R and Q. Say that R is *strictly disjoint from* Q if $\boldsymbol{r} \sqcup \boldsymbol{q} = \square$, i.e. if they have no common subject-matter (apart from $\square$), and say that R is a *strict remainder of* P *from* Q if R is a remainder of P from Q that is strictly disjoint from Q.

We tie together the internal and external characterizations of remainder:

Theorem 10 *Suppose P and Q are regular propositions within a remaindered space with $Q \leqslant P$ and $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{q}$. Then:*

(i) There exists a strict remainder of P from Q iff for each $q \in Q$, $q \sqcup \boldsymbol{r} \in P$;
(ii) If there exists a strict remainder of P from Q then (a) it is identical to $P - Q$, (b) it is contained in every other remainder of P from Q, and (c) it contains every part R of P strictly disjoint from Q and hence is the conjunction of all such parts.

Proof (i) First suppose R is a strict remainder of P from Q; and take any $q \in Q$. Then for some $r \in R$, $q \sqcup r \in P$. Since $R \leqslant P$ by lemma 7(i), $r \sqsubseteq \boldsymbol{p}$ and, since R is a strict remainder, r overlaps no $q \in Q$ and hence does not overlap $\boldsymbol{q}$. But then $r \sqsubseteq \boldsymbol{r} = \boldsymbol{p} - \boldsymbol{q} \sqsubseteq \boldsymbol{p}$ and so $q \sqcup r \sqsubseteq q \sqcup \boldsymbol{r} \sqsubseteq \boldsymbol{p}$ and $q \sqcup \boldsymbol{r} \in P$.

Now suppose $q \sqcup \boldsymbol{r} \in P$ for each $q \in Q$. Let $\Pi = \{(q_p, r_p)\colon p \in P\} \cup \{(q, \boldsymbol{r})\colon q \in Q\}$. Then it is evident that $\Pi_1 = Q$. Also, $\Pi_{1,2} \subseteq P$, given that $q_p \sqcup r_p = p \in P$ and that $q \sqcup \boldsymbol{r} \in P$ for each $q \in Q$, and $P \sqsubseteq \Pi_{1,2}$ given that, for each $p \in P$, $q_p \sqcup r_p = p$. Moreover, $R = \Pi_2$ is strictly disjoint from Q and so R is a strict remainder of P from Q.

Note that Π_2, with Π defined above, is identical to $P - Q$; and so we have also established:

(1) if there exists a strict remainder of P given Q then $P - Q$ is such a remainder.

(ii)(a) & (b). This will follow from the following additional facts:

(2) If R is a remainder of P given Q then $R \geqslant P - Q$.

Pf. Suppose R is a remainder of P given Q. For one direction, take $r \in R$. Then for some $q \in Q$ and $p \in P$, $r \sqcup q = p$. But $q \sqsubseteq q_p$; and so, $r \sqsupseteq p - q \sqsupseteq p - q_p = r_p \in P - Q$. For the other direction, we note that, by lemma 9(i), $\boldsymbol{p} - \boldsymbol{q}$ is the maximal element of $P - Q$; and so we need to show that, for some $r \in R$, $\boldsymbol{p} - \boldsymbol{q} \sqsubseteq r$. Since R is a remainder, there is a $r \in R$ and a $q \in Q$ for which $r \sqcup q = \boldsymbol{p}$. Given $q \sqsubseteq \boldsymbol{q} \sqsubseteq \boldsymbol{p}$, $r \sqcup \boldsymbol{q} = \boldsymbol{p}$; and so $\boldsymbol{p} - \boldsymbol{q} \sqsubseteq r$.

(3) If R is a strict remainder of P given Q then $R \leqslant P - Q$.

Pf. Suppose R is a strict remainder of P given Q. For one direction, take $r \in R$. Then $r \sqsubseteq \boldsymbol{p}$ and, since R is strictly disjoint from Q, r is disjoint from $\boldsymbol{q}$; and so $r \sqsubseteq \boldsymbol{p} - \boldsymbol{q}$ and $\boldsymbol{p} - \boldsymbol{q} \in P - Q$ by lemma 9(i). For the other direction, take $r \in P - Q$. Then r is of the form r_p for $p \in P$. Since R is a remainder, there is a $q \in Q$ and a $r' \in R$ for which $r' \sqcup q = p$. If $r' \sqsubseteq r$ we are done. So suppose not $r' \sqsubseteq r$. Then some non-null $r'' \sqsubseteq r'$ is disjoint from $r = r_p = p - q_p$ and, since $r'' \sqsubseteq r' \sqsubseteq p$, r'' overlaps q_p, contrary to the supposition that R is a strict remainder.

(ii) (c) Take a part R of P strictly disjoint from Q. Suppose first that $r \in R$. Since $R \leqslant P$, $r \sqsubseteq \boldsymbol{p}$. But r is disjoint from $\boldsymbol{q}$ and so $r \sqsubseteq \boldsymbol{p} - \boldsymbol{q} \in P - Q$. Now take $p - \boldsymbol{q} \in P - Q$ for $p \in P$. Since $R \leqslant P$, $r \sqsubseteq p$ for some $r \in R$. But r is disjoint from $\boldsymbol{q}$ and so $r \sqsubseteq p - \boldsymbol{q}$. □

Some notes. (1) P will be a coordinated conjunction of $P - Q$ and Q but not, in general, a straight conjunction. In order for P to be a straight conjunction it should be required, not merely that $q \sqcup \boldsymbol{r} \in P$ for each $q \in Q$, but also that $q \sqcup r_p \in P$ for each $p \in P$ and $q \in Q$. (2) The condition under (i) is equivalent to: for each $q \in Q$, there is an $r \sqsubseteq \boldsymbol{r}$ for which $q \sqcup r \in P$. For $q \sqcup r \sqsubseteq q \sqcup \boldsymbol{r} \sqsubseteq \boldsymbol{p}$ and so, by Convexity, $q \sqcup \boldsymbol{r} \in P$ if $q \sqcup r \in P$. (3) The condition under (i) may fail. A simple example is with $P = \{\text{pq}\}$ (corresponding to the conjunction p $\wedge$ q) and $Q = \{\text{p, q}\}$ (corresponding to the disjunction p $\vee$ q) within the canonical space. Q is then a part of P but $\boldsymbol{p} - \boldsymbol{q}$ is the null state and so there is no state contained in $\boldsymbol{p} - \boldsymbol{q}$ that can fuse with either p or q to give back pq.

Jaeger [10] has considered the question of when (to state it in our terms) $R \wedge Q = R' \wedge Q$ implies $R = R'$; and Humberstone ([9], p. 62 et seq.) has considered the related question of when $(R \wedge Q) - Q = R$. From the above result, we see:

Corollary 11 *(i)* $(R \wedge Q) - Q = R$ *given that* $\boldsymbol{r}$ *is disjoint from* $\boldsymbol{q}$*;*
(ii) $R \wedge Q = R' \wedge Q$ *implies* $R = R'$*, given that* $\boldsymbol{r}$ *and* $\boldsymbol{r}'$ *are both disjoint from* $\boldsymbol{q}$*.*

Proof (i) Suppose $P = (R \wedge Q)$ with $\boldsymbol{r}$ disjoint from $\boldsymbol{q}$. Then R is a strict remainder of P from Q; and so, by (ii)(a) of the theorem, $R = (R \wedge Q) - Q$
(ii) Suppose that $R \wedge Q = R' \wedge Q$ with $\boldsymbol{r}$ and $\boldsymbol{r}'$ both disjoint from $\boldsymbol{q}$. Then $(R \wedge Q) - Q = R$ and $(R' \wedge Q) - Q = R'$ by (i) above; and so $R = R'$. □

In the special case in which the subtracted proposition is determinate, the remainder has an especially simple form:

Corollary 12 *Suppose that* P *is a regular proposition* $\{p_1, p_2, ...\}$ *and that* $Q \leqslant P$ *is a determinate proposition* $\{q\}$ *within a remaindered space. Then* $(P - Q) = \{p_1 - q, p_2 - q, ...\}$ *and* $P = (P - Q) \wedge Q$*.*

Proof Given that $Q \leqslant P$ and that P is regular, $q \sqcup (\boldsymbol{p} - \boldsymbol{q}) = q \sqcup (\boldsymbol{p} - q) = \boldsymbol{p} \in P$ and so, by (i) and (ii)(a) of theorem 10, $P - Q$ is a strict remainder. But given that Q is the determinate proposition $\{q\}$, $(P - Q) = \{p_1 - q, p_2 - q, ...\}$ and the only coordinated conjunction of Q and $P - Q$ is $(P - Q) \wedge Q$.
□

Let us briefly discuss the corresponding notion of *disjunctive* remainder. With conjunctive remainder we remove a conjunct and thereby obtain a lesser proposition, while with disjunctive remainder we remove a disjunct and thereby obtain a stronger proposition. Given propositions P and Q, we may take $P -_{\text{d}} Q$ to be $\{p \in P$: p

disjoint from $\boldsymbol{q}$} (and we might now write $P -_{\mathrm{c}} Q$ in place of $P - Q$ to bring out the contrast with $P -_{\mathrm{d}} Q$). $R = P -_{\mathrm{d}} Q$ will be a regular proposition when P and Q are regular but with $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{q}$. In place of the notion of coordinated conjunction, we have the notion of expanded disjunction, where the expanded disjunction of P and Q may contain states of the form $p' \sqcup q'$ for $p' \sqsubseteq$ some p in P and $q' \sqsubseteq$ some q in Q in addition to the members of P and of Q. When it comes to the bilateral case, we might, as before, define $\boldsymbol{P} -_{\mathrm{c}} \boldsymbol{Q}$, where $\boldsymbol{P} = (P, P')$ and $\boldsymbol{Q} = (Q, Q')$, to be $(P -_{\mathrm{c}} Q, \sim(P -_{\mathrm{c}} Q))$ but we might also take it to be $(P -_{\mathrm{c}} Q, P' -_{\mathrm{d}} Q')$, though without any assurance that the resulting proposition will be 'well-behaved'.

We finally turn to the topic of weak conjunctive remainder, for which the strict disjointness condition is relaxed. Given propositions P and Q with $Q \leqslant P$, let $P / Q = \{r$: for some $q \in Q, r \sqcup q \in P\}$. (When not $Q \leqslant P$, we may identify P / Q with $P \sqcup (P \nabla Q)$). Say that two propositions P and Q are *disjoint* (as opposed to *strictly* disjoint) if they have no non-trivial proposition as a common conjunctive part.

Theorem 13 *For regular verifiable propositions P and Q with $Q \leqslant P$, $R = P / Q$ is the weakest remainder of P from Q and it is a regular proposition for which* $\boldsymbol{r} = \boldsymbol{p}$ *and which, within a remaindered space, is disjoint from Q.*

Proof We suppose P and Q are regular verifiable propositions with $Q \leqslant P$ and that $R = P \sqcup Q$.

(1) R is a regular verifiable proposition.
Pf. Since P is verifiable, it contains a member $p \in P$. Since $Q \leqslant P$, $q \sqcup p$ for some $q \in Q$. But then $q \sqcup p = p \in P$; and so $q \in R$ and R is verifiable.
Further, $\boldsymbol{r} = \bigsqcup R \in R$. For, given any $r_{\mathrm{i}} \in R$, there is a $q_{\mathrm{i}} \in Q$ for which $r_{\mathrm{i}} \sqcup q_{\mathrm{i}} \in P$. But $\bigsqcup r_{\mathrm{i}} \sqcup \bigsqcup q_{\mathrm{i}} = \bigsqcup (r_{\mathrm{i}} \sqcup q_{\mathrm{i}})$; $\bigsqcup (r_{\mathrm{i}} \sqcup q_{\mathrm{i}}) \in P$ since P is regular and $\bigsqcup q_{\mathrm{i}} \in Q$ since Q is regular; and so $\boldsymbol{r} = \bigsqcup r_{\mathrm{i}} \in R$.
(b) Suppose $r \sqsubseteq r' \sqsubseteq r^+$, with $r, r^+ \in R$. Then for some $q, q^+ \in Q$, $r \sqcup q$, $r^+ \sqcup q^+ \in P$. But $r \sqcup q \sqsubseteq r' \sqcup q \sqsubseteq \boldsymbol{p}$. By P regular, $r' \sqcup q \in P$; and so $r' \in R$.

(2) R is a remainder of P from Q.
Pf. Let $\Pi = \{(r, q): q \in Q, r \sqcup q \in P\}$. Then clearly $\Pi_1 = R$ and $\Pi_2 = Q$ since, for any $q \in Q$, there is a $p \in P$ for which $q \sqsubseteq p$ and so for which $q \sqcup p \in P$. Moreover, $R \wedge_\Pi Q = P$. For clearly, $R \wedge_\Pi Q \subseteq P$. Now suppose $p \in P$. Then $p \sqsupseteq q$ for some $q \in Q$ and so $p = p \sqcup q \in R \wedge_\Pi Q$. Hence $P \subseteq R \wedge_\Pi Q$.

(3) R is the weakest remainder.
Pf. Suppose that r' is a remainder. So $r' \wedge_\Pi Q = P$ for some differentiated content Π. Take any member r of r'. Then $r \sqcup q \in P$ for some $q \in P$ and so $r \in P \sqcup Q$.

(4) $\boldsymbol{r} = \boldsymbol{p}$.
Pf. $\boldsymbol{p} \sqcup \boldsymbol{q} = \boldsymbol{p}$ with $\boldsymbol{q} \in Q$. So $\boldsymbol{p} \in R$ and $\boldsymbol{p} \sqsubseteq \boldsymbol{r}$. Also, for each $r \in R$, $r \sqsubseteq \boldsymbol{p}$ and so $\boldsymbol{r} \sqsubseteq \boldsymbol{p}$.

(5) R is disjoint from Q in a remaindered space.
Pf. $\boldsymbol{p} - \boldsymbol{q} \in R$ since $(\boldsymbol{p} - \boldsymbol{q}) \sqcup \boldsymbol{q} = \boldsymbol{p}$. Suppose now that Q and R have in common a non-trivial part T. Then $\boldsymbol{p} - \boldsymbol{q} \sqsupseteq t$ for some non-null state $t \in T$

and also $t \sqsubseteq q \sqsubseteq \boldsymbol{q}$ for some $q \in Q$. But then t is a common non-null part of $\boldsymbol{q}$ and $\boldsymbol{p} - \boldsymbol{q}$, which is impossible. □

The contrast between the two forms of remainder may be brought out by considering the case in which P is subtracted from P. As is readily verified, $P - P$ is the completely trivial proposition $T_\square = \{\square\}$ while $P \,/\, P$ is the trivial proposition $\{p\colon p \sqsubseteq \boldsymbol{p}\}$. More generally:

Corollary 14 *Suppose that $P - Q$ exists and that $Q \leqslant P$. Then $P \,/\, Q = [P - Q, \boldsymbol{p}]$.*

Proof Take any $r \in P \sqcup Q$. Then for some $r' \in P - Q$, $r' \sqsubseteq r$ since $P - Q \leqslant P \,/\, Q$ by (ii)(b) of theorem 10. But $r \sqsubseteq \boldsymbol{p}$. Hence $r' \sqsubseteq r \sqsubseteq \boldsymbol{p}$; and so $r \in [P - Q, \boldsymbol{p}]$. Take now $r \in [P - Q, \boldsymbol{p}]$. Then for some $r' \in P - Q$, $r' \sqsubseteq r \sqsubseteq \boldsymbol{p}$. But $r' \sqcup q \in P$ for some $q \in Q$; and so $r \sqcup q \in P$ and $r \in P \sqcup Q$. □

Subject-Matter

Recall that we take the subject-matter $\boldsymbol{p}$ of a proposition P to be $\bigsqcup P$ and, when P is a regular verifiable proposition, $\boldsymbol{p} \in P$ and is the maximal verifier of P.

Let us summarize the previous results on subject-matter:

Theorem 15 *Let P_1, P_2, ... be regular verifiable propositions:*

(i) When $P = P_1 \wedge P_2 \wedge \ldots, \boldsymbol{p} = \bigsqcup \boldsymbol{p}_i$
(ii) When $P = P_1 \bigtriangledown P_2 \bigtriangledown \ldots, \boldsymbol{p} = \bigsqcap \boldsymbol{p}_i$
(iii) When $P = P_1 \vee P_2 \vee \ldots, \boldsymbol{p} = \bigsqcup \boldsymbol{p}_i$
(iv) When $P = P_1 \vartriangle P_2 \vartriangle \ldots, \boldsymbol{p} \sqsubseteq \bigsqcap \boldsymbol{p}_i$
(v) When $R = P - Q$ exists, $\boldsymbol{r} = \boldsymbol{p} - \boldsymbol{q}$
(vi) When $R = P \,/\, Q$, $\boldsymbol{r} = \boldsymbol{p}$.

In other words, the subject-matter of a conjunction or a disjunction is the sum of the subject-matter of the conjuncts or disjuncts, the subject matter of the common part of some propositions is the common part of the their subject-matters; the subject-matter of the common disjunctive part of some propositions is a part of their common subject matter; the subject-matter of the least remainder is the result of subtracting the subject matter of the subtracted proposition from the given proposition; and the subject-matter of the weakest remainder is the same as the subject-matter of the given proposition.

Given a bilateral proposition $\boldsymbol{P} = (P, P')$, there will, of course, be the *positive subject-matter* $\boldsymbol{p}$ and the *negative subject-matter* (or *anti-matter*) $\boldsymbol{p}'$. But we may also associate with $\boldsymbol{P}$ the *comprehensive* subject-matter $\boldsymbol{p}_{\pm} = \boldsymbol{p} \sqcup \boldsymbol{p}'$ and the *differentiated* subject-matter $\boldsymbol{p}_{+/-} = (\boldsymbol{p}, \boldsymbol{p}')$. Given the regular verifiable propositions $\boldsymbol{P}_1 = (P_1, P_1')$, $\boldsymbol{P}_2 = (P_2, P_2')$, ..., the comprehensive subject matter $\boldsymbol{p}_{\pm} = \boldsymbol{p} \sqcup \boldsymbol{p}'$ of $\boldsymbol{P} = \boldsymbol{P}_1 \wedge \boldsymbol{P}_2 \wedge \ldots = (P_1 \wedge P_2 \wedge \ldots, P_1' \vee P_2' \vee \ldots)$ will be the fusion of the comprehensive subject-matters $\boldsymbol{p}_{1\pm} = \boldsymbol{p}_1 \sqcup \boldsymbol{p}_1'$, $\boldsymbol{p}_{2\pm} = \boldsymbol{p}_2 \sqcup \boldsymbol{p}_2'$, ... of $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ..., while the differentiated subject-matter $\boldsymbol{p}_{+/-} = (\boldsymbol{p}, \boldsymbol{p}')$ of $\boldsymbol{P}$ will be fusion of the differentiated subject-matters $\boldsymbol{p}_{1+/-} = (\boldsymbol{p}_1, \boldsymbol{p}_1')$, $\boldsymbol{p}_{2+/-} = (\boldsymbol{p}_2, \boldsymbol{p}_2')$, ... of $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, ... The comprehensive subject-matter of $\neg\boldsymbol{P}$ will be the same as the comprehensive subject-matter of $\boldsymbol{P}$, while the differentiated subject-matter $(\boldsymbol{p}', \boldsymbol{p})$ of $\neg\boldsymbol{P}$ will be the 'reverse' of the differentiated subject-matter $(\boldsymbol{p}, \boldsymbol{p}')$ of $\boldsymbol{P}$.

We may *restrict* a unilateral proposition P to some subject-matter $\boldsymbol{s} \sqsubseteq \boldsymbol{p}$. We take the *restriction* $P^{\mathbf{s}}$*of* P *to* $\boldsymbol{s}$ to be $\{p \sqcap \boldsymbol{s}: p \in P\}$, which we may also denote by $P \sqcap \boldsymbol{s}$. Thus $P^{\mathbf{s}}$ is the common content $P \nabla \{\boldsymbol{s}\}$ of the proposition P and the associated subject-matter content $\{\boldsymbol{s}\}$. So, for example, within the canonical space when $P = \{\text{pq, pr}\}$ and $\boldsymbol{s} = \text{qr}$, $P^{\mathbf{s}} = \{\text{q, r}\}$.

We may also *expand* a proposition P to some subject-matter $\boldsymbol{s} \sqsupseteq \boldsymbol{p}$. We take the *expansion* $P^{\mathbf{s}}$*of* P *to* $\boldsymbol{s}$ to be $\{P'$: for some $p \in P$, $p \sqsubseteq p' \sqsubseteq \boldsymbol{s}\}$. Thus when $P = \{\text{p, q}\}$ and $\boldsymbol{s} = \text{pqr}$, $P^{\mathbf{s}} = \{\text{p, q, pq, pr, qr, pqr}\}$.

In general, we may have some subject-matter $\boldsymbol{s}$ for which neither $\boldsymbol{s} \sqsubseteq \boldsymbol{p}$ nor $\boldsymbol{p} \sqsubseteq \boldsymbol{s}$. In this case, we would like to restrict by $\boldsymbol{s} \sqcap \boldsymbol{p}$ and expand by $\boldsymbol{s}$. We therefore take $P^{\mathbf{s}}$ - the *conformation of* P *to* $\boldsymbol{s}$ - to be $[P \sqcap \boldsymbol{s}, \boldsymbol{s}]$. Thus when $P = \{\text{pq, pr}\}$ and $\boldsymbol{s} = \text{qrs}$, $P^{\mathbf{s}} = \{\text{q, r, qr, qs, rq, rs, qrs}\}$.

Lemma 16 *Suppose P is a regular verifiable proposition and* $\boldsymbol{s}$ *some subject-matter. Then:*

(i) $P^{\mathbf{s}}$ *is a regular verifiable proposition;*
(ii) $\boldsymbol{p}^{\mathbf{s}} = \boldsymbol{s}$;
(iii) $P^{\mathbf{s}} = P \sqcap \boldsymbol{s}$ *and* $P^{\mathbf{s}} \leqslant P$ *when* $\boldsymbol{s} \sqsubseteq \boldsymbol{p}$
(iv) $P^{\mathbf{s}} = [P, \boldsymbol{s}]$ *and* $P \leqslant P^{\mathbf{s}}$ *when* $\boldsymbol{p} \sqsubseteq \boldsymbol{s}$
(v) $P^{\mathbf{ss}} = P^{\mathbf{s}}$
(vi) $P^{\mathbf{s}} \geqslant P^{\mathbf{t}}$ *iff* $\boldsymbol{s} \sqsupseteq \boldsymbol{t}$
(vii) $P^{\mathbf{s}} = (P^{\mathbf{s} \sqcap \mathbf{p}})^{\mathbf{s}} = (P^{\mathbf{s} \sqcup \mathbf{p}})^{\mathbf{s}}$

Proof (i) Since P is verifiable, it contains a verifier p. But then $p \sqcap \boldsymbol{s} \sqsubseteq p \sqcap \boldsymbol{s} \sqsubseteq \boldsymbol{s}$; and so $p \sqcap \boldsymbol{s} \in P^{\mathbf{s}}$ and $P^{\mathbf{s}}$ is verifiable. Moreover, since $P^{\mathbf{s}}$ is of the form $[P \sqcap \boldsymbol{s}, \boldsymbol{s}]$, it is automatically regular.

(ii) Evident from the definition of $\boldsymbol{p}^{\mathbf{s}}$.

(iii) $P^{\mathbf{s}} = [P \sqcap \boldsymbol{s}, \boldsymbol{s}]$. So clearly, $P \sqcap \boldsymbol{s} \subseteq P^{\mathbf{s}}$. For the other direction, suppose $q \in P^{\mathbf{s}}$. Then for some $p \in P$, $p \sqcap \boldsymbol{s} \sqsubseteq q \sqsubseteq \boldsymbol{s}$. Since $\boldsymbol{s} \sqsubseteq \boldsymbol{p}$, $q \sqsubseteq \boldsymbol{p}$ and so $p \sqcup q \in P$. But $(p \sqcup q) \sqcap \boldsymbol{s} = (p \sqcap \boldsymbol{s}) \sqcup (q \sqcap \boldsymbol{s}) = (p \sqcap \boldsymbol{s}) \sqcup q = q$ and so $q \in P \sqcap \boldsymbol{s}$. Given $P \sqcap \boldsymbol{s} \subseteq P^{\mathbf{s}}$ and $P^{\mathbf{s}} \subseteq P \sqcap \boldsymbol{s}$, it follows that $P^{\mathbf{s}} = P \sqcap \boldsymbol{s}$. Moreover, it is evident that $P^{\mathbf{s}} = P \sqcap \boldsymbol{s} \leqslant P$.

(iv) $P^{\mathbf{s}} = [P \sqcap \boldsymbol{s}, \boldsymbol{s}]$. But given $\boldsymbol{p} \sqsubseteq \boldsymbol{s}$, $P \sqcap \boldsymbol{s} = P$ and so $[P \sqcap \boldsymbol{s}, \boldsymbol{s}] = [P, \boldsymbol{s}]$. Moreover, it is evident that $P \leqslant [P, \boldsymbol{s}] = P^{\mathbf{s}}$.

(v) By (ii), $\boldsymbol{p}^{\mathbf{s}} = \boldsymbol{s}$. So it suffices to show that $P^{\mathbf{s}} = P$ when $\boldsymbol{p} = \boldsymbol{s}$. But in this case, it follows by (iv) that $P^{\mathbf{s}} = [P, \boldsymbol{s}] = [P, \boldsymbol{p}] = P$.

(vi) Suppose $P^{\mathbf{s}} \geqslant P^{\mathbf{t}}$. Then $\boldsymbol{p}^{\mathbf{s}} \sqsupseteq \boldsymbol{p}^{\mathbf{t}}$. But by (ii), $\boldsymbol{p}^{\mathbf{s}} = \boldsymbol{s}$ and $\boldsymbol{p}^{\mathbf{t}} = \boldsymbol{t}$; and so $\boldsymbol{s} \sqsupseteq \boldsymbol{t}$. Now suppose $\boldsymbol{s} \sqsupseteq \boldsymbol{t}$. Take $q \in P^{\mathbf{s}}$. Then for some $p \in P$, $p \sqcap \boldsymbol{s} \sqsubseteq q \sqsubseteq \boldsymbol{s}$. Since $\boldsymbol{s} \sqsupseteq \boldsymbol{t}$, $p \sqcup \boldsymbol{s} \sqsupseteq p \sqcup \boldsymbol{t}$; and so $q \sqsupseteq p \sqcup \boldsymbol{t} \in P^{\mathbf{t}}$. For the remaining clause in the definition of $\geqslant$, we need to show $\boldsymbol{p}^{\mathbf{s}} \sqsupseteq \boldsymbol{p}^{\mathbf{t}}$. But this follows from the fact that $\boldsymbol{p}^{\mathbf{s}} = \boldsymbol{s}$, $\boldsymbol{p}^{\mathbf{t}} = \boldsymbol{t}$, and $\boldsymbol{s} \sqsupseteq \boldsymbol{t}$.

(vii) $P^{\mathbf{s}} = [P \sqcap \boldsymbol{s}, \boldsymbol{s}] = [P \sqcap \boldsymbol{s} \sqcap \boldsymbol{p}, \boldsymbol{s}] = [[P \sqcap \boldsymbol{s} \sqcap \boldsymbol{p}, \boldsymbol{s} \sqcap \boldsymbol{p}], \boldsymbol{s}] = (P^{\mathbf{s} \sqcap \mathbf{p}})^{\mathbf{s}}$. Also, $(P^{\mathbf{s} \sqcup \mathbf{p}})^{\mathbf{s}} = [[P \sqcap (\boldsymbol{s} \sqcup \boldsymbol{p}), (\boldsymbol{s} \sqcup \boldsymbol{p})], \boldsymbol{s}] = [[P, \boldsymbol{s} \sqcup \boldsymbol{p}], \boldsymbol{s}] = \{q \sqcap \boldsymbol{s}: p \sqsubseteq q \sqsubseteq \boldsymbol{s} \sqcup \boldsymbol{p}$ for some $p \in P\} = \{r: p \sqcap \boldsymbol{s} \sqsubseteq r \sqsubseteq \boldsymbol{s}$ for some $p \in P\}$ (i.e. $P^{\mathbf{s}}$) since if $r = q \sqcap \boldsymbol{s}$ for $p \sqsubseteq q \sqsubseteq \boldsymbol{s} \sqcup \boldsymbol{p}$ for some $p \in P$ then $p \sqcap \boldsymbol{s} \sqsubseteq r \sqsubseteq \boldsymbol{s}$ and if $p \sqcap \boldsymbol{s} \sqsubseteq r \sqsubseteq \boldsymbol{s}$ for some $p \in P$ then, setting $q = r \sqcup p$, $p \sqsubseteq q \sqsubseteq \boldsymbol{s} \sqcup \boldsymbol{p}$ and $q \sqcap \boldsymbol{s} = (r \sqcup p) \sqcap \boldsymbol{s} = (r \sqcap \boldsymbol{s}) \sqcup (p \sqcap \boldsymbol{s}) = r \sqcup (p \sqcap \boldsymbol{s}) = r$. □

The last result ((vii)) says that the conformation of a proposition to some subject-matter can be seen both as the product of a successive restriction and expansion or as the product of a successive expansion and a restriction.

In the bilateral case, we might define the *restriction of* $\boldsymbol{P}^{\mathbf{s}} = (P, P')$ *to* $\boldsymbol{s}$, for $\boldsymbol{p} \sqsubseteq \boldsymbol{s}$ and $\boldsymbol{p}' \sqsubseteq \boldsymbol{s}$, to be $(P^{\mathbf{s}}, P^*)$, where $P^* = \{q \in P': q \sqsubseteq \boldsymbol{s}\}$. For suitable choices of $\boldsymbol{P}$ and $\boldsymbol{s}$, it might then be shown that $\boldsymbol{P}^{\mathbf{s}}$ is well-behaved when $\boldsymbol{P}$ is well-behaved.

In a remaindered space, we can define, for any given subject-matter $\boldsymbol{s}$, its *anti-matter* (▪-$\boldsymbol{s}$), which we designate as $\overline{\boldsymbol{s}}$. Just as we may restrict the content P to $\boldsymbol{s} \sqcap \boldsymbol{p}$, we may also restrict it to $\overline{\boldsymbol{s}} \sqcap \boldsymbol{p}$. Thus $P^{\mathbf{s} \sqcap \mathbf{p}} = \{\boldsymbol{s} \sqcap p: p \in P\}$ and $P^{\overline{\mathbf{s}} \sqcap \mathbf{p}} = \{\overline{\boldsymbol{s}} \sqcap p: p \in P\}$. It is helpful to coordinate the respective contents $P^{\mathbf{s} \sqcap \mathbf{p}}$ and $P^{\overline{\mathbf{s}} \sqcap \mathbf{p}}$. To this end, we define the *differentiated restricted content* $P^{\mathbf{s}, \overline{\mathbf{s}}}$ to be $\{(\boldsymbol{s} \sqcap p, \overline{\boldsymbol{s}} \sqcap p): p \in P\}$; and, more generally, $P^{\mathbf{s}, \mathbf{t}} = \{(\boldsymbol{s} \sqcap p, \boldsymbol{t} \sqcap p): p \in P\}$.

Lemma 17 *Suppose that* P *is a regular verifiable proposition in a remaindered space and that* $\Pi = P^{\mathbf{s}, \overline{\mathbf{s}}}$. *Then* Π *is a regular differentiated content for which* $P^{\mathbf{s}} \wedge_{\Pi} P^{\overline{s}} = P$.

Proof P contains the maximal verifier $\boldsymbol{p}$ and so Π contains the maximal verifier $(\boldsymbol{s} \sqcap \boldsymbol{p}, \overline{\boldsymbol{s}} \sqcap \boldsymbol{p})$. Now suppose $(\boldsymbol{s} \sqcap p, \overline{\boldsymbol{s}} \sqcap p) \sqsubseteq \pi \sqsubseteq (\boldsymbol{s} \sqcap \boldsymbol{p}, \overline{\boldsymbol{s}} \sqcap \boldsymbol{p})$ for $p \in P$ (to show $\pi \in \Pi$). Then π is of the form (π_1, π_2) with $\boldsymbol{s} \sqcap p \sqsubseteq \pi_1 \sqsubseteq \boldsymbol{s} \sqcap \boldsymbol{p}$ and $\overline{\boldsymbol{s}} \sqcap p \sqsubseteq \pi_2 \sqsubseteq \overline{\boldsymbol{s}} \sqcap \boldsymbol{p}$. Let $p^* = \pi_1 \sqcup \pi_2$. Since $\boldsymbol{s} \sqcap p \sqsubseteq \pi_1$ and $\overline{\boldsymbol{s}} \sqcap p \sqsubseteq \pi_2$, $p = (\boldsymbol{s} \sqcup \overline{\boldsymbol{s}}) \sqcap p = (\boldsymbol{s} \sqcap p) \sqcup (\overline{\boldsymbol{s}} \sqcap p) \sqsubseteq p^*$. Since $\pi_1 \sqsubseteq \boldsymbol{s} \sqcap \boldsymbol{p}$ and $\pi_2 \sqsubseteq \overline{\boldsymbol{s}} \sqcap \boldsymbol{p}$, $\pi_1 \sqcup \pi_2 \sqsubseteq (\boldsymbol{s} \sqcap \boldsymbol{p}) \sqcup (\overline{\boldsymbol{s}} \sqcap \boldsymbol{p}) = \boldsymbol{p}$. Hence $p \sqsubseteq p^* \sqsubseteq \boldsymbol{p}$ and $p^* \in P$. But $\boldsymbol{s} \sqcap p^* = \boldsymbol{s} \sqcap (\pi_1 \sqcup \pi_2) = (\boldsymbol{s} \sqcap \pi_1) \sqcup (\boldsymbol{s} \sqcap \pi_2) = \pi_1$, given that $\pi_1 \sqsubseteq \boldsymbol{s}$ and $\pi_2 \sqsubseteq \overline{\boldsymbol{s}}$; and similarly $(\overline{\boldsymbol{s}} \sqcap p^*) = \pi_2$. Hence $\pi = (\pi_1, \pi_2) = (\boldsymbol{s} \sqcap p^*, \overline{\boldsymbol{s}} \sqcap p^*) \in \Pi$. □

It should be clear that $\Pi_1 = P^{\mathbf{s}}$ and that $\Pi_2 = P^{\overline{\mathbf{s}}}$. Moreover, $\Pi_{1,2} = \{(\boldsymbol{s} \sqcap p) \sqcup (\overline{\boldsymbol{s}} \sqcap p): p \in P\} = \{(\boldsymbol{s} \sqcup \overline{\boldsymbol{s}}) \sqcap p: p \in P\} = P$.

Note that, since $P^{\mathbf{s}}$ is strictly disjoint from $P^{\overline{s}}$, it follows that $P^{\overline{s}} = P - P^{\mathbf{s}}$, just as one might have thought.

Various other notions of aboutness might also be considered. Let us focus on the notion of *partial* aboutness. Given the subject-matter $\boldsymbol{s}$ and the proposition P, P is said to be *partly about* $\boldsymbol{s}$ if $\boldsymbol{p}$ and $\boldsymbol{s}$ overlap. We have the following partial compositionality results for partial aboutness:

Lemma 18 *For regular verifiable propositions* P *and* Q*, the following are equivalent:*

(i) $P \wedge Q$ *is partly about* $\mathbf{s}$*;*
(ii) $P \vee Q$ *is partly about* $\mathbf{s}$*;*

(iii) P *is partly about* $\boldsymbol{s}$ *or* Q *is partly about* $\boldsymbol{s}$.

Proof Suppose $R = P \wedge Q$ and $R' = P \vee Q$. Then $\boldsymbol{r} = \boldsymbol{r}' = \boldsymbol{p} \sqcup \boldsymbol{q}$; and so $P \wedge Q$ is partly about $\boldsymbol{s}$ iff $P \vee Q$ is partly about $\boldsymbol{s}$.

Now suppose $R = P \wedge Q$ is partly about $\boldsymbol{s}$. Then $\boldsymbol{s}$ overlaps with $\boldsymbol{r} = \boldsymbol{p} \sqcup \boldsymbol{q}$ and so, by Overlap, $\boldsymbol{s}$ overlaps with $\boldsymbol{p}$ or with $\boldsymbol{q}$ and P or Q is partly about $\boldsymbol{s}$.

Now suppose P is partly about $\boldsymbol{s}$ (the case in which Q is partly about $\boldsymbol{s}$ is similar). Then $\boldsymbol{s}$ overlaps with $\boldsymbol{p}$ and hence with $\boldsymbol{r}$ for $R = P \wedge Q$ and $P \wedge Q$ is partly about $\boldsymbol{s}$. □

This and the subsequent results extend straightforwardly to infinitary conjunctions and disjunctions. The result may also be extended to negation by talking of bilateral propositions instead of unilateral propositions. Say that the bilateral proposition $\boldsymbol{P} = (P, P')$ is *positively* (*negatively*) *partially about* the subject-matter $\boldsymbol{s}$ if P (resp. P') is partially about $\boldsymbol{s}$. Then from the previous lemma it immediately follows that:

Theorem 19 *For regular non-vacuous propositions* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *and subject-matter* $\mathbf{s}$*:*

- *(i)* $\neg\boldsymbol{P}$ *is positively (negatively) partially about* $\mathbf{s}$ *iff* $\boldsymbol{P}$ *is negatively (positively) partially about* $\mathbf{s}$*;*
- *(ii)* $\boldsymbol{P} \wedge \boldsymbol{Q}$ *is positively (negatively) partially about* $\mathbf{s}$ *iff* $\boldsymbol{P}$ *or* $\boldsymbol{Q}$ *is positively (negatively) partially about* $\mathbf{s}$*; and*
- *(iii)* $\boldsymbol{P} \vee \boldsymbol{Q}$ *is positively (negatively) partially about* $\mathbf{s}$ *iff* $\boldsymbol{P}$ *or* $\boldsymbol{Q}$ *is positively (negatively) partially about* $\mathbf{s}$.

Now say that the bilateral proposition $\boldsymbol{P}$ is *partially about* subject-matter $\boldsymbol{s}$ if it is positively or negatively partially about $\boldsymbol{s}$. Note that $\boldsymbol{P} = (P, P')$ will be partially about $\boldsymbol{s}$ just in case $\boldsymbol{s}$ overlaps with the combined subject-matter $\boldsymbol{p} \sqcup \boldsymbol{p}'$ of $\boldsymbol{P}$. From the theorem, we obtain the more general compositionality result.

Corollary 20 *For regular nonvacuous propositions* $\boldsymbol{P}$ *and* $\boldsymbol{Q}$ *and subject-matter* $\mathbf{s}$*:*

- *(i)* $\neg\boldsymbol{P}$ *is partially about* $\mathbf{s}$ *iff* $\boldsymbol{P}$ *is partially about* $\mathbf{s}$*;*
- *(ii)* $\boldsymbol{P} \wedge \boldsymbol{Q}$ *is partially about* $\mathbf{s}$ *iff* $\boldsymbol{P}$ *or* $\boldsymbol{Q}$ *is partially about* $\mathbf{s}$*; and*
- *(iii)* $\boldsymbol{P} \vee \boldsymbol{Q}$ *is partially about* $\mathbf{s}$ *iff* $\boldsymbol{P}$ *or* $\boldsymbol{Q}$ *is partially about* $\mathbf{s}$.

Ground

We adopt the following definitions from [3]. For verifiable propositions $P_1, P_2, ...$ and Q, we say:

$P_1, P_2, ...$ *weakly* (*fully*)*grounds* Q - in symbols, $P_1, P_2, ... \leqslant Q$ - if $P_1 \wedge P_2 \wedge ... \leqslant_d Q$;

P *weakly partially grounds* Q - in symbols, $P Q$ - if for some verifiable $P_1, P_2, ..., P, P_1, P_2, ...$ weakly grounds Q;

$P_1, P_2, ...$ *strictly (fully)grounds* Q - in symbols, $P_1, P_2, ... < Q$ - if $P_1, P_2, ...$ *weakly grounds* Q and Q does not weakly partially ground any of the propositions $P_1, P_2, ...$;

P *strictly partially grounds* Q - in symbols, $P \prec Q$ - if P weakly grounds Q but Q does not weakly ground P.

Although I have given these definitions for the case of unilateral propositions, they are readily extended to the case of bilateral propositions. Note also that it will follow from these definitions that the resulting notions of ground conform to the pure logic of ground, as laid down in [3].

Lemma 21 *For regular verifiable propositions P and Q, the following are equivalent:*

(i) P *weakly partially grounds* Q
(ii) P, P' *weakly grounds* Q *for some regular verifiable proposition* P'
(iii) P, Q *weakly grounds* Q
(iv) $(P \wedge Q) \vee Q = Q$
(v) $\boldsymbol{p} \sqsubseteq \boldsymbol{q}$.

Proof It is evident that (iii) ⇒ (ii) and that (ii)⇒ (i). Three cases remain: (i) ⇒ (v). Suppose P weakly partially grounds Q, so $P \wedge P_1 \wedge P_2 ... \leqslant_d Q$ for regular verifiable propositions $P_1, P_2, ...$. Select $p_1 \in P_1, p_2 \in P_2, ...$. Then $\boldsymbol{p} \sqcup p_1 \sqcup p_2 \sqcup ... \in P \wedge P_1 \wedge P_2 ... \subseteq Q$; and so $\boldsymbol{p} \sqsubseteq \boldsymbol{p} \sqcup p_1 \sqcup p_2 \sqcup ... \sqsubseteq \boldsymbol{q}$.

(v) ⇒ (iv) & (iv) ⇒ (iii). By lemma I.13, conditions (iii) and (iv) are equivalent and so it suffices to show (v) ⇒ (iii). To this end, suppose $\boldsymbol{p} \sqsubseteq \boldsymbol{q}$ and take a $p \sqcup q \in P \wedge Q$ with $p \in P$ and $q \in Q$. We wish to show $p \sqcup q \in Q$. But $q \sqsubseteq p \sqcup q \sqsubseteq \boldsymbol{p} \sqcup q \sqsubseteq \boldsymbol{p} \sqcup \boldsymbol{q} \sqsubseteq \boldsymbol{q}$ (given $\boldsymbol{p} \sqsubseteq \boldsymbol{q}$); and so by Q regular, $p \sqcup q \in Q$. □

The most remarkable equivalence here is that weak partial ground ($P \preceq Q$) amounts simply to inclusion of subject-matter ($\boldsymbol{p} \sqsubseteq \boldsymbol{q}$). Note that it follows immediately from this equivalence that P will strictly partially ground Q ($P \prec Q$) just in case there is a proper inclusion of subject-matter ($\boldsymbol{p} \sqsubsetneq \boldsymbol{q}$).

Theorem 22 *For regular verifiable* $P_1, P_2, ...$ *and Q, the following are equivalent:*

(i) $P_1, P_2, ...$ *strictly grounds* Q
(ii) $P_1, P_2, ...$ *weakly grounds* Q *and each of* P_i *strictly partially grounds* Q
(iii) $P_1, P_2, ...$ *weakly grounds* Q *and each* $\boldsymbol{p}_i \sqsubsetneq \boldsymbol{q}$

Proof The equivalence of (i) and (ii) is immediate from the definitions; and the equivalence of (ii) and (iii) follows from the criterion for strict partial ground. □

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